Strategic Bidding for Multiple Price-Maker Hydroelectric Producers

Gregory Steeger[a], Steffen Rebennack[b]

[a]*Colorado School of Mines, Division of Economics and Business, Golden, CO 80401, gsteeger@mymail.mines.edu*
[b]*Colorado School of Mines, Division of Economics and Business, Golden, CO 80401, srebenna@mines.edu*

Abstract

In a market comprised of multiple price-maker firms, the payoff each firm receives depends not only on one's own actions but also on the actions of the other firms. This is the defining characteristic of a non-cooperative economic game. In this paper, we ask: What is the revenue-maximizing production schedule for multiple price-maker hydroelectric producers competing in a deregulated, bid-based market? In every time stage, we seek a set of bids such that, given all other price-maker producers' bids, no price maker can improve (increase) their revenue by changing their bid, *i.e.*, a pure strategy Nash-Cournot equilibrium. From a theoretical game theory perspective, the analysis on the underlying non-cooperative game is lacking. Specifically, existing approaches are not able to detect when multiple equilibria exist and consider any equilibrium found optimal. In our approach, we create interpolations for each price maker's best response function using mixed-integer linear programming formulations within a dynamic programming framework. In the presence of multiple Nash equilibria, when one exists, our approach finds the equilibrium that is Pareto optimal. If a Pareto-optimal Nash equilibrium does not exist, we use a tailored bargaining algorithm to determine a unique solution. To illustrate some of the finer details of our method, we present three examples and a case study on Honduras.

Keywords: Dynamic programming, mixed-integer linear programming, optimal production scheduling, revenue maximization, bargaining, bidding problem, hydroelectric producer, Nash equilibrium, Nash-Cournot, Pareto, potential game, price-maker

1. Introduction

Our growing reliance on renewable energy sources in general, and on hydropower in particular, demands we develop more efficient ways to model its use in real-world markets. A significant amount of power is allocated through the day-ahead electricity market. Typical day-ahead electricity markets can be described by the following three phases: bidding, clearing and pricing. In the bidding phase, producers submit price and/or quantity bids. In the clearing phase, the market operator (utility company) adjusts the market-clearing price to meet demand while minimizing costs. And finally, in the pricing phase, producers are paid for the amount of energy they dispatch, according to the market-clearing price.

Hydroelectric producers possess the unique ability to store energy (water) for use in future periods, in which it may generate more profit. The ability to store water at almost no cost creates

Accepted at IIE Transactions

an opportunity cost for producers, associated with producing energy today versus saving it for use in a future stage. This creates a non-trivial coupling between the decisions made in every stage.

More often than not, practitioners assume that producers' bids will have no impact on the market-clearing price. Because the demand for electricity is relatively inelastic, Kelman et al. [25] argue that changes in aggregate supply will alter the marker-clearing price. Similarly, Pousinho et al. [37] argue that many electricity market structures are vulnerable to firms that possess some amount of market power, so-called price-makers. Market power is the ability to influence prices in one's favor. In this paper, we assume the market consists of both price-maker hydroelectric producers and price-taker thermal producers, producers whose bids have no impact on the market-clearing price.

When a competitive market consists of multiple price-maker producers, the revenue for each producer depends on the bids from all other price-maker producers [3]. In this case, all of the price-maker producers act simultaneously to maximize their individual revenue. The scenario is usually modeled via non-cooperative economic game theory in which the practitioner seeks to find a set of bids such that, given all other price-maker producers' bids, no price-maker producer can improve (increase) their revenue by changing their bid. Such a condition constitutes a Nash equilibrium [33].

In this paper, we study the bidding problem for multiple price-maker hydroelectric producers competing in a deregulated, bid-based market. We first identify areas in the literature in which the state-of-the-art on economic game theory has not been applied. We then develop a Dynamic Programming (DP) formulation of the problem which attempts to mitigate some of the shortcomings in the literature. Our methodology determines a price maker's revenue-maximizing bids, subject to all other price makers' best responses to those bids, through a Mixed-Integer Linear Programming (MILP) formulation. When this is done for all price makers, the answer defines a Nash equilibrium. Each price-maker's MILP incorporates interpolations of the other price makers' best response functions. The interpolation used to approximate a price maker's best response function is created through solving an MILP formulation that is based on discrete bids and reservoir levels from the other price makers. In cases with multiple equilibria, our method selects a Pareto-optimal Nash equilibrium, when one exists, or uses a tailored bargaining algorithm to determine the equilibrium that should be implemented. Due to the shape of each price makers' revenue function, it is possible for a continuum of Nash equilibria to exist. In this application, we are the first to detect a continuum and suggest a way through which to select a unique equilibrium.

Our contributions to the body of literature on hydro producer modeling are twofold.

- First, we list recommendations for future research directions in the area of economic game theory applied to the bidding problem (Section 2).
- Second, we introduce a DP method that finds a revenue-maximizing hydro operations bidding schedule for multiple price-maker producers. Our method distinguishes between multiple Nash equilibria and either selects the Pareto-optimal equilibrium, when one exists, or utilizes a tailored bargaining approach to select a unique equilibrium (Section 4).

The remainder of this paper is arranged as follows. In Section 2, we review the current literature, present some general observations, and compare our approach to existing approaches. Next, in Section 3, we review the fundamentals of game theory that are relevant to our problem. We present our methodology in Section 4, demonstrate the utility of our approach through illustrative examples and a case study in Section 5, and list areas for future research in Section 6. Finally, Section 7 presents our findings and conclusions.

2. Current Literature and Shortcomings

The problem in which producers submit bids to the day-ahead electricity market is termed the *bidding problem* [13]. Table 1 lists several references that contribute to solving the bidding problem for multiple price-maker producers. All of the references listed in the table rely on mathematical models that aim to find Nash equilibria. The table is by no means exhaustive, but is a representative sample of approaches used in the literature.

Table 2 further classifies the references from Table 1 based on the solution approach that is used to find a Nash equilibrium and whether or not they (i) discuss the existence of an equilibrium, (ii) discuss multiple equilibria, (iii) classify the equilibrium that is found, (iv) make the state space discrete, (v) discuss which equilibrium is best, and (vi) discuss mixed-strategy equilibria. Tables 1 and 2 list both thermal and hydroelectric producers to illustrate how the problem has been studied in depth for thermal producers, but not for hydroelectric producers. See Steeger et al. [46] for a survey on this problem. Because of key differences between thermal producers and hydroelectric producers (namely storage), the models used for thermal producers cannot be readily applied to hydroelectric producers.

Table 1: References on the bidding problem for multiple price-maker producers: Overview

Reference	Year	Overview
Scott and Read [43]	1996	Evaluate regulated vs. deregulated markets – dual DP with Cournot duopolies
Ramos et al. [39]	1999	Max welfare – Mathematical Program with Equilibrium Constraints (MPEC)
Borenstein et al. [4]	1999	Nash-Cournot measure to evaluate market power vs. concentration measures
Barquin et al. [1]	2000	Measure the value of water in a competitive (deregulated) environment
Hobbs et al. [23]	2000	Profit max for each firm – MPEC solved via an interior point algorithm
Kelman et al. [25]	2001	Measure market power – use stochastic DP to approximate future cost functions
Hobbs [21]	2001	Bilateral market – mixed linear complementarity problem
Garcia et al. [16]	2001	Evaluate regulatory policies via a Bertrand oligopoly with Markovian states
Bushnell [6]	2003	Characterization of Cournot equilibrium in hydrothermal oligopoly
Villar and Rudnick [48]	2003	Extend [25] for strategic thermal and hydro firms
de la Torre et al. [10]	2004	Seek maximum revenue for all price makers – simulate market
Xian et al. [49]	2004	Max social welfare – nonlinear complementarity model solved via a heuristic
Barroso et al. [2]	2006	Max the price-makers' revenue via Binary expansion
Hasan et al. [20]	2008	Seek all pure equilibria via a MILP & a supply function equilibrium model
Hasan and Galiana [19]	2008	Part II of [20] – select best equilibrium based off risk-dominance
Nanduri and Das [31]	2009	Seek optimal bidding strategies via reinforcement algorithm and payoff matrices
Kannan et al. [24]	2011	Seek optimal bidding strategies in a two-settlement framework
Pozo and Contreras [38]	2011	Max price-makers' profit – EPEC used to find all pure strategy equilibria
Yang et al. [50]	2012	Seek optimal bidding strategies via payoff matrices and polynomial equations

From reviewing the literature we observe that the research on the problem for hydroelectric producers is outdated, in that few provide an in-depth analysis from a game theory perspective.

Table 2: References on the bidding problem for multiple price-maker producers: Classification. A [✓] denotes a reference that discusses the associated attribute in general, but not in the specific context of the problem.

Reference	Hydro	Solution Approach	Existence	Multiple NE	Classify	Discrete Strategy Space	"Best" NE	Mixed Strategy
Scott and Read [43]	✓	Calculus	✓	✓	-	-	[✓]	-
Ramos et al. [39]	✓	MPEC	-	-	-	-	-	-
Borenstein et al. [4]	-	Calculus	-	✓	-	-	✓	-
Barquin et al. [1]	✓	Calculus	[✓]	[✓]	-	-	[✓]	[✓]
Hobbs et al. [23]	-	Iteration	✓	[✓]	-	-	-	[✓]
Kelman et al. [25]	✓	Iteration	-	-	-	-	-	-
Hobbs [21]	-	LCP	✓	✓	-	-	-	-
Garcia et al. [16]	✓	Markov chains	-	-	-	-	-	-
Bushnell [6]	✓	LCP	-	-	-	-	-	-
Villar and Rudnick [48]	✓	Iteration	-	-	-	-	-	-
de la Torre et al. [10]	-	Simulation	✓	✓	✓	-	✓	[✓]
Xian et al. [49]	-	Calculus	-	-	-	-	-	-
Barroso et al. [2]	-	Iteration	✓	✓	-	✓	✓	-
Hasan et al. [20]	-	MILP	✓	✓	-	✓	-	[✓]
Hasan and Galiana [19]	-	MILP	✓	✓	✓	✓	✓	[✓]
Nanduri and Das [31]	-	Iteration	-	✓	-	✓	-	-
Kannan et al. [24]	-	LCP	✓	✓	-	-	-	-
Pozo and Contreras [38]	-	Iteration	✓	✓	-	✓	-	-
Yang et al. [50]	-	Algebra	✓	✓	-	✓	✓	✓

Kannan et al. [24] present one exception in which the authors provide a thorough analysis, but focus solely on thermal producers that sell energy in both a forward market and an uncertain real-time market.

The first area in which the most recent advances in game theory can be applied is the existence of equilibria. Some practitioners state the general criteria for which an equilibrium exists, *e.g.*, they state that mixed strategy equilibria are guaranteed to exist for finite games, yet they only seek pure strategy equilibria in a finite game and do not discuss when these are guaranteed to exist [10, 2]. Others, neither guarantee existence nor state criteria for existence, but state that they believe in most cases that pure strategy equilibria exist [22]. Under a given set of sufficient conditions, approaches that solve a Linear Complementarity Problem (LCP) not only guarantee existence but also guarantee a unique solution [7, 21]. Hasan et al. [20] solve a MILP and identify the conditions necessary for the existence of pure strategy Nash equilibria, but only when the strategy space is discrete and the game is finite. Using a discretization scheme may actually eliminate Nash equilibria in the approximated continuous game and may even lead to the case in which no pure strategy Nash equilibrium exists [2, 50]; *cf.* Section 3. These observations leads to our first specific remark regarding the bidding problem for price-maker hydroelectric producers.

***Remark* 1.** *Practitioners need to provide a thorough analysis on the existence of pure strategy Nash equilibria when the strategy space is continuous and the revenue function is non-concave, e.g., discontinuous.*

Second, practitioners need to provide further analyses when multiple equilibria exist. With multiple equilibria, many authors consider any equilibrium found, to be optimal. In this particular application, it is even possible for a continuum of Nash equilibria to exist (see Example 2) and this is typically not discussed in the literature. There are, however, a few papers that present ideas on how to find all equilibria and/or how to determine the "best" equilibrium. Pozo and Contreras [38] find all pure Nash equilibria in a Stackelberg game via a stochastic Equilibrium Problem with Equilibrium Constraints (EPEC). In de la Torre et al. [10], based on the results of a simulation, six bidding strategies are selected as possible options for each of three price makers. A market-clearing formulation is solved for each of the 216 possible scenarios and the solutions are examined for Nash equilibria. Dominated strategies are removed and a total of seven pure strategy Nash equilibria are found and characterized. Yang et al. [50] present a method, based off the payoff matrix approach and polynomial equations, to find all Nash equilibria in an electricity market with multiple price maker thermal producers. Hasan et al. [20] and Hasan and Galiana [19] present a paper in two parts. The first finds all pure strategy Nash equilibria and the second selects the "best" one.

In the case when solution methods may result in multiple equilibria and researchers want to select the one that is "best," different criteria may be used. Examples include defining the "best" equilibrium as the one with the highest price [4, 2], the one with the lowest risk [20], or the one that is payoff dominant, *i.e.*, Pareto optimal [50]. de la Torre et al. [10] remove dominated equilibria (based off profits), but in the end, are still left with multiple equilibria to select from. The above observations precipitate

***Remark* 2.** *Practitioners need to develop a method that determines if and when multiple equilibria exist when the strategy space is continuous.*

***Remark* 3.** *In the presence of multiple Nash equilibria, the research community needs to come to a consensus on the criteria used to determine which Nash equilibrium is "best."*

***Remark* 4.** *In the presence of multiple Nash equilibria, practitioners need to develop a method that computes the "best" one.*

Third, in the absence of multiple simplifying assumptions [50], we observe that

***Remark* 5.** *Current approaches do not seek mixed-strategy Nash equilibria.*

The last area in which advances can be made regarding the game-theoretic aspects of this problem is in uncertainty. In this context, many parameters and/or data can be considered uncertain including the reservoir inflows [1, 25, 16] and bids from other market participants [24], thus making the game stochastic. Consequently, we present one final remark.

***Remark* 6.** *Practitioners need to consider the inherent uncertainty involved and model the problem as a stochastic game.*

Our approach is a first attempt at mitigating some of the shortcomings associated with the remarks above for the bidding problem for multiple price-maker hydroelectric producers. However, a lot remains to be done; *cf.* see Section 6. Of the papers listed in Table 1, only six seek a Nash equilibrium (in every stage) for hydroelectric producers. Of those, four utilize a Dynamic Programming (DP) scheme similar to ours, they are: Scott and Read [43], Barquin et al. [1], Kelman et al. [25], and Villar and Rudnick [48]. Table 3 portrays the differences and similarities between our approach and the approaches in these four papers.

Table 3: Comparisons of how different aspects of the bidding problem are modeled

Dimension	Approach	Reference(s)
Revenue or profit function	Concave overestimation	Scott and Read [43]
	Differentiable	Barquin et al. [1]
	Quadratic	Kelman et al. [25]
	Nonlinear differentiable	Villar and Rudnick [48]
	Stepwise linear	Our approach
Inflow uncertainty	Discrete distributions	Barquin et al. [1], Kelman et al. [25]
	Probability curves	Scott and Read [43]
	Not considered	Villar and Rudnick [48], our approach
Price-Taker competitors' bids	Known (variable cost)	Kelman et al. [25], Villar and Rudnick [48], Our approach
	N/A	Scott and Read [43], Barquin et al. [1]
Time-dependency	DDP	Scott and Read [43]
	DP	Barquin et al. [1], Villar and Rudnick [48], Our approach
	SDP	Kelman et al. [25]
Demand	Demand curves	Scott and Read [43], Barquin et al. [1] Villar and Rudnick [48]
	Deterministic	Kelman et al. [25], our approach
Bid structure	Cournot	Scott and Read [43], Barquin et al. [1], Kelman et al. [25], Villar and Rudnick [48] Our approach
Nash equilibrium found via	Algebraic manipulation	Scott and Read [43]
	Unspecified method	Barquin et al. [1]
	Iterative approach	Kelman et al. [25], Villar and Rudnick [48]
	Discretization & optimization	Our approach

3. Fundamentals

Our convention is to denote parameters with uppercase letters, decision variables with lowercase letters, and vectors of parameters or decision variables with boldface characters. We define the following:

Indices and Sets:

$n = 1, \ldots, N$ price-maker producers

$i = 1, \ldots, I$ price-makers' reservoirs

$j = 1, \ldots, J$ price-taker thermal producers

$k = 1, \ldots, K$ breakpoints of $R_t\left(\sum_{n=1}^{N} e_{nt}\right)$

$t = 1, \ldots, T$ time periods

$-n$ all price makers, excluding price-maker n

$\mathbb{I}_n$ set of reservoirs operated by price-maker n

$\mathbb{U}_{in}$ reservoirs immediately upstream of reservoir i, for price-maker n

Parameters:

D_t demand in market, in stage t [GWh]

$\overline{G}_j$ quantity bid for producer j [GWh]

C_j price bid for producer j [\$/GWh]

E'_{kt} k^{th} breakpoint of $R_t\left(\sum_{n=1}^{N} e_{nt}\right)$, in stage t [GWh]

π^d_{kt} market-clearing price, of revenue segment k, in stage t [\$/Gwh]

$\underline{V}_{in}$ minimum storage storage level for reservoir i, for price-maker n [hm^3]

$\overline{V}_{in}$ maximum storage level for reservoir i, for price-maker n [hm^3]

V_{in1} initial storage level for reservoir i, for price-maker n [hm^3]

$\overline{U}_{in}$ water turbine outflow capacity for reservoir i, for price-maker n [hm^3]

A_{int} inflow for reservoir i, for price-maker n, in stage t [hm^3]

ρ_{in} production coefficient for reservoir i, for price-maker n [GWh/hm^3]

Functions:

$R_t\left(\sum_{n=1}^{N} e_{nt}\right)$ aggregate revenue function for all price makers, in stage t [\$]

$R_{nt}(e_{1t}, \ldots, e_{Nt})$ immediate revenue function, for price-maker n, in stage t [\$]

$\beta_{nt}(\mathbf{V}_{nt}, \mathbf{V}_{-nt})$ price-maker n's revenue from stage t to T [\$]

$\gamma_{nt}(e_{-nt}, \mathbf{V}_{nt}, \mathbf{v}_{-n,t+1})$ price-maker n's best response function, in stage t [\$]

Binary Decision Variables:

ϕ_{kt} indicator variable used to determine the market-clearing price [-]

Decision Variables:

g_j quantity of energy produced (sold) for price-taker producer j [GWh]

π^d_t market-clearing price, in stage t [\$/GWh]

e_{nt} total hydro production, for price-maker n, in stage t [GWh]

e_{int} hydro production from reservoir i, for price-maker n, in stage t [GWh]

s_{int} spillage from reservoir i, for price-maker n, in stage t [hm^3]

v_{int} volume of reservoir i, for price-maker n, at the beginning of stage t [hm^3]

$\mathbf{v}_{nt}$ vector of reservoir volumes, for price-maker n, at the beginning of stage t [hm^3]

In our application, the players are the price-maker hydro producers; all price makers possess perfect information, *i.e.,* for each move, each player has full knowledge of all of their competitors' previous moves; each price maker can choose to produce anywhere in the range between zero and their capacity, in other words, the action space is continuous; and each player's payoff is the revenue they receive, in that stage.

Since, the price-maker producers simultaneously decide how much to bid (produce) and then the price is chosen to clear the market by the ISO, we are specifically interested in the Cournot model. In simultaneous games, the most common solution concept is that of a *Nash equilibrium* [33]. A set of strategies defines a Nash equilibrium, if and only if each player's move is a best response to the moves played by his rivals. Kelman et al. [25] define a Nash equilibrium as a "standoff" situation in which no firm can unilaterally increase its revenue by changing its production. In this paper, we are only concerned with pure-strategy equilibria and not mixed-strategy equilibria; thus, any reference to a Nash equilibrium refers to a pure-strategy equilibrium. In a multi-stage problem, we could either satisfy the conditions for a Nash equilibrium over the entire time horizon or in each stage. Since the decisions made today will impact future production, we chose the latter, which is much more restrictive.

Our problem is repeated over an indefinite time horizon, but we approximate the problem by assuming a time horizon of length T. The game is not a *T-stage repeated game*, because it is dynamic and in a *repeated game*, past plays cannot influence the feasible actions of payoffs in the current stage. In our game, the decisions made today are dependent on the water reservoir levels, which are determined in the previous stage. The water reservoir state variables capture the effect of the previous stage's decisions on the current stage.

To be certain that the game we are playing is understood, we define it in standard terms used in Economic game theory [30]. For N players, let

$z_{nt} \in \mathbb{S}_{nt}(\mathbf{V}_{nt})$ denote a *strategy* for player n in stage t, based on $\mathbf{V}_{nt}$, and

$u_{nt}(z_{1t}, \ldots, z_{nt})$ denote the *payoff function* (or revenue function) for player n in stage t.

With this notation in place

$$\Gamma_t := [N, \{\mathbb{S}_{nt}(\mathbf{V}_{nt})\}, \{u_{nt}(\cdot)\}], \tag{1}$$

is the normal form representation of the game. Since this game is repeated, Γ_t is defined for every stage t. Though each strategy z_{nt} depends on $\mathbf{V}_{nt}$, we omit this to avoid overcomplicating the notation, *i.e.,* we use z_{nt} as opposed to $z_{nt}(\mathbf{V}_{nt})$. In our application, $\mathbb{S}_{nt}(\mathbf{V}_{nt})$ is a polyhedron defined by a constraint set. In other words, the *strategy space* for each player, in each stage, is *compact*. More specifically,

$$\begin{aligned}
\mathbb{S}_{nt}(\mathbf{V}_{nt}) := \Big\{ e_{nt}, v_{i,n,t+1} : \; & e_{nt} = \sum_{i \in \mathbb{I}_n} e_{int}, \\
& 0 \le \frac{e_{int}}{\rho_{in}} \le \overline{U}_{in} \quad \forall\, i \in \mathbb{I}_n, \\
& v_{i,n,t+1} = V_{int} + A_{int} - \frac{e_{int}}{\rho_{in}} - s_{int} + \sum_{m \in \mathbb{U}_{in}} \left(\frac{e_{mnt}}{\rho_{mn}} + s_{mnt} \right) \quad \forall\, i \in \mathbb{I}_n, \\
& \underline{V}_{in} \le v_{i,n,t+1} \le \overline{V}_{in} \quad \forall\, i \in \mathbb{I}_n, \\
& s_{int} \ge 0 \quad \forall\, i \in \mathbb{I}_n \Big\}
\end{aligned}$$

The constraints and bounds in the above strategy space are explained in greater detail in Section 4. Note that the strategy space $\mathbb{S}_{nt}(\mathbf{V}_{nt})$ is connected to the previous stage in the problem through the state parameters V_{int}. All past decisions are represented by these state parameters, which are decision variables in stage $t-1$. This means that the strategy space and a Nash equilibrium, in stage t, depend on the decisions made in the previous stage. For this reason, each strategy is termed a *Markov strategy*. Thus, over the entire time horizon, we seek a *Markov Perfect Equilibrium (MPE)*.

If we let $-n$ denote all price-makers except price-maker n, the payoff function is given as

$$u_{nt}(z_{nt}, z_{-nt}) := R_{nt}(e_{nt}, e_{-nt}) \qquad \forall\, n, t, z_{nt} \in \mathbb{S}_{nt}(\mathbf{V}_{nt}), z_{-nt} \in \mathbb{S}_{-nt}(\mathbf{V}_{-nt}).$$

We discuss specifics regarding the $R_{nt}(\cdot)$ functions in Section 4. A strategy space in stage t, for all N players $\{z^*_{nt}\}$, $z^*_{nt} \in \mathbb{S}_{nt}(\mathbf{V}_{nt})$ is a Nash equilibrium if

$$u_{nt}(z^*_{nt}, z^*_{-nt}) \geq u_{nt}(z_{nt}, z^*_{-nt}) \qquad \forall\ n,\ z_{nt} \in \mathbb{S}_{nt}(\mathbf{V}_{nt}). \tag{2}$$

Each stage can either yield a unique Nash equilibrium, a finite set of Nash equilibria, or a continuum of Nash equilibria. For instances with multiple Nash equilibria, the practitioner must determine which is "best" and may utilize economic bargaining theory to do so. In bargaining, one seeks a unique Nash equilibrium through solving what is termed the bargaining problem. In the bargaining problem, players determine how to split some portion of a good [26]. In the split, player one receives ζ percent and player two receives $1 - \zeta$ percent of the good, with $\zeta \in [0, 1]$. Nash, in [32] and [33], provides a solution to this problem, through an axiomatic derivation. The "Nash solution" must satisfy a set of axioms and the solution can be found through solving

$$\begin{aligned} \max_{(x,y)\in\mathbb{F}}\ & (U_1(x) - D_x)(U_2(y) - D_y) \\ \text{s.t. }\ & U_1(x) \geq D_x \\ & U_2(y) \geq D_y, \end{aligned}$$

in which $U_1(x)$ and $U_2(y)$ are the utility functions for each price maker, D_x and D_y are the disagreement payoffs (or status quo utilities), and $\mathbb{F}$ is the feasible set of players' actions. D_x and D_y are the utilities each price maker would expect if they cannot reach an agreement with the other price maker. We refer to the solution of the above formulation as the "Nash solution." The solution favors the player(s) with the most bargaining strength or their relative ability to exert influence over other players [28].

4. Methodology

Having built the necessary foundation, we now present our methodology.

4.1. Revenue Function

To determine the shape of each price maker's revenue function, we examine their maximum-revenue problem, the day-ahead electricity market, and the way in which the market-clearing price is determined, for any stage t. Price maker n seeks to maximize revenue, or mathematically

$$\max\ R_{nt}(e_{1t}, \ldots, e_{Nt}) := \pi^d_t \cdot e_{nt}.$$

We model all price-maker's bids simultaneously and utilize logical MILP constraints to determine the market-clearing price and resulting revenue for each price maker. To determine how the revenue function for price maker n changes with different values of e_{nt}, we assume that each price taker bids its variable operating cost and capacity, which is their optimal decision in a market consisting solely of price takers [18]. In the day-ahead electricity market, the operator satisfies demand while minimizing cost and the market-clearing price, π^d_t, is determined based on the value

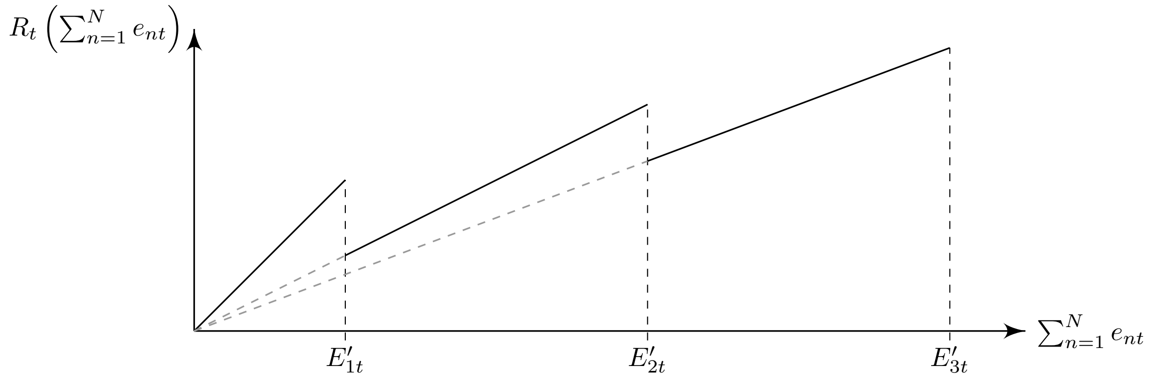

Figure 1: Joint revenue function

of $\sum_{n=1}^{N} e_{nt}$. The market-clearing formulation, for a fixed bid quantity from each price maker, E_{nt}, is given as

$$\begin{aligned} \min \ & \sum_{j=1}^{J} C_j g_j \\ \text{s.t.} \ & \sum_{j=1}^{J} g_j = D_t - \sum_{n=1}^{N} E_{nt} && (\pi_t^d) \\ & 0 \leq g_j \leq \overline{G}_j \qquad \forall j. \end{aligned}$$

The solution to the market-clearing formulation yields both the market-clearing price π_t^d, and the market-clearing dispatch. The market-clearing price is found via the dual of the demand constraint, whereas the set of $g_j,\ j = 1, \ldots, J$, such that $g_j > 0$ defines the market-clearing dispatch. In this construct, the market-clearing price is the highest price bid from the set of price-taker producers that are part of the market-clearing dispatch. Regardless of their bids, all producers that are part of the market-clearing dispatch are paid the market-clearing price π_t^d.

Knowledge of each price-taker j's bid allows for the population of parameters C_j and $\overline{G}_j$ in the market-clearing formulation and helps determine the market-clearing price and the shape of the aggregate price maker's revenue function. We plot the revenue as a function of the aggregated total of all the price makers' quantity bids, $R_t\left(\sum_{n=1}^{N} e_{nt}\right)$, in Figure 1. The revenue curve assumes a sawtooth shape, because as the price makers' bids collectively increase, the most costly price-taker producer that is part of the market-clearing dispatch "drops out." As price-taker producers are forced out of the market-clearing dispatch, the market-clearing price (slope of the revenue curve) decreases.

Because the market-clearing price is constant over fixed intervals of $\sum_{n=1}^{N} e_{nt}$, we can model how it changes using binary variables and "big $\mathcal{M}$ constraints." Assuming π_{kt}^d (a parameter) is the market-clearing price (or slope) for line segment k, we maximize the revenue, in stage t, for price maker ω using the following objective function and constraints:

$$R_{\omega t}(e_{1t}, \ldots, e_{Nt}) = \max \psi \tag{3}$$

$$\text{s.t.} \sum_{n=1}^{N} e_{nt} \leq E'_{kt} + \mathcal{M}\phi_{kt} \qquad \forall \ k \tag{4}$$

$$\psi \leq \pi^d_{kt} e_{\omega t} + \mathcal{M}\phi_{kt} \qquad \forall \ k \tag{5}$$

$$\sum_{k=1}^{K} (1 - \phi_{kt}) = 1 \tag{6}$$

$$\phi_{kt} \in \{0, 1\} \qquad \forall \ k. \tag{7}$$

We use the big $\mathcal{M}$ constraints shown in (4) and (5) to determine the segment in which the cumulative production lies. Constraints (6) and (7) ensure that exactly one of the ϕ variables equals zero and consequently force exactly one of the K constraints in both (4) and (5) to bind. Since we are maximizing ψ, since the π^d_{kt} parameters decrease as the index k increases, and since exactly one of the ϕ variables equals zero, it will be desirable to select the π^d_{kt} with the smallest k index. Constraint (4) restricts us from choosing too small of an index based on the cumulative production. Together, these constraints guarantee the correct market-clearing price, based off cumulative production, and thus the correct revenue for price maker ω. Since, $e_{int} \leq \overline{U}_{in}\rho_{in}$ and π^d_{1t} is the largest possible market-clearing price, for each of price-maker n's reservoirs, we set

$$\mathcal{M} := \pi^d_{1t} \sum_{n=1}^{N} \sum_{i=1}^{I_n} \overline{U}_{in}\rho_{in}.$$

Our formulation for finding price maker ω's revenue is similar to the one found in de la Torre et al. [9], however ours has a simpler objective function and fewer constraints.

4.2. *Existence of Nash Equilibria*

Mas-Colell et al. [30] state that a pure strategy Nash equilibrium exists, if for all $n = 1, \ldots, N$, (i) $\mathbb{S}_{nt}(\mathbf{V}_{nt})$ is a nonempty, convex, and compact subset of some Euclidean space $\mathbb{R}^m$, and (ii) $u_{nt}(z_{nt}, z_{-nt})$ is continuous in (z_{nt}, z_{-nt}) and quasiconcave in z_{nt}. In our application, (i) is satisfied, but (ii) is not.

A pure strategy equilibrium can still exist with discontinuous and non-quasiconcave payoff functions. Dasgupta and Maskin [8] show that a pure strategy equilibrium is guaranteed to exist if the strategy space is compact and the payoff functions are *upper semi-continuous*. These characterizations are important, because typically, finding a Nash equilibrium is a fixed point problem [29]. An upper semi-continuous function is a function with no downward jumps. Stated formally [15], $u_{nt}(z_{nt}, z_{-n,t})$ on $\mathbb{S}_{nt}(\mathbf{V}_{nt})$ is upper semi-continuous at z_{nt}, if, for any sequence z^k_{nt} converging to z_{nt},

$$\limsup_{k \to +\infty} \ u_{nt}(z^k_{nt}, z_{-n,t}) \leq u_{nt}(z_{nt}, z_{-n,t}).$$

Unfortunately, each player's revenue function has downward jumps and is lower semi-continuous as opposed to upper semi-continuous. $u_{nt}(z_{nt}, z_{-n,t})$ on $\mathbb{S}_{nt}(\mathbf{V}_{nt})$ is lower semi-continuous at z_{nt}, if, for any sequence z_{nt}^k converging to z_{nt},

$$\liminf_{k\to+\infty} u_{nt}(z_{nt}^k, z_{-n,t}) \geq u_{nt}(z_{nt}, z_{-n,t}).$$

Mallozzi [29] suggests that a class of economic games known as *potential games* are the "link between optimization and game theory." Potential games are attractive because, under predetermined conditions pure strategy equilibria are guaranteed to exist [5]. Similarly, Panicucci et al. [34] discuss how a specific class of Nash equilibrium problems in which the payoff function is both differentiable and convex can equivalently be written as optimization problems. However, potential games are less restrictive. Economic games in which a *potential function* exists are termed potential games.

According to Branzei et al. [5], $\Gamma_t = [N, \{\mathbb{S}_{nt}(\mathbf{V}_{nt})\}, \{u_{nt}(\cdot)\}]$ is a potential game if there exists a function $P : \prod_{n=1}^{N} \mathbb{S}_{nt}(\mathbf{V}_{nt}) \to \mathbb{R}$ such that

$$\begin{aligned} u_{nt}(z_{nt}, z_{-nt}) - u_{nt}(z'_{nt}, z_{-nt}) &= P(z_{nt}, z_{-nt}) - P(z'_{nt}, z_{-nt}) \\ &\qquad \forall\ n,\ z_{nt} \in \mathbb{S}_{nt}(\mathbf{V}_{nt}),\ z'_{nt} \in \mathbb{S}_{nt}(\mathbf{V}_{nt}), z_{-nt} \in \mathbb{S}_{-nt}(\mathbf{V}_{-nt}). \end{aligned}$$

In our case, the potential function is simply the joint revenue function, or

$$P(z_{nt}, z_{-nt}) = R(e_{nt}, e_{-nt}).$$

Potential games are of interest, because, under certain conditions, optimizing a potential function results in a solution that is a Nash equilibrium [29]. According to Mallozzi, when firms jointly maximize their potential function, the resulting solution is a Nash equilibrium. Mallozzi also states that the set of all strategy profiles that maximize P is a subset of the set of Nash equilibria for the game. For our problem, this would mean that any point that maximizes the joint revenue is a Nash equilibrium, which is not true (see Figure 5). In this example, the continuum of points from ($e_1 = 150, e_2 = 50$) to ($e_1 = 200, e_2 = 0$) maximize joint revenue, while only the continuum of points from ($e_1 = 150, e_2 = 50$) to ($e_1 = 168.88, e_2 = 31.12$) are Nash equilibria. Though Mallozzi does not clearly state it, based on Reny [41], Philippe [36] and Kukushkin [27], we believe he is assuming the potential functions are upper semi-continuous. To our knowledge, the existence of a Nash equilibrium is only guaranteed if the payoff functions or potential functions are upper semi-continuous. Thus, since this is not the case for our payoff function or potential function, we cannot rely on existing literature to prove the existence of a Nash equilibrium.

With continuous strategy spaces there are well known examples in which a pure strategy equilibrium exists [11, 17, 12], a pure strategy equilibrium does not exist while a mixed-strategy equilibrium does [42], and in which neither a pure strategy nor a mixed-strategy equilibrium exists [44].

If each price maker can only select from a discrete set of bids, then a pure strategy Nash equilibrium may or may not exist. For example, Figure 2 shows the total aggregate revenue and best responses for a scenario with two price makers. On top is the joint revenue function (Nash equilibrium depicted by dashed line) and on bottom is a plot of discrete best responses for each price maker. Price-maker one's best response is black, whereas price-maker two's best response is gray. We keep this convention for the remainder of the plots shown below. A pure strategy

Nash equilibrium exists at any and all intersections of the players' best response curves. The best response functions are computed and plotted for eight discrete bids, and though a Nash equilibrium exists with continuous bids, by selecting this discrete set of bids we do not find it. Barroso et al. [2] assumes each price maker can only select from a discrete set of bids, and consequently admit that it is possible for no pure strategy Nash equilibrium to exist for their problem.

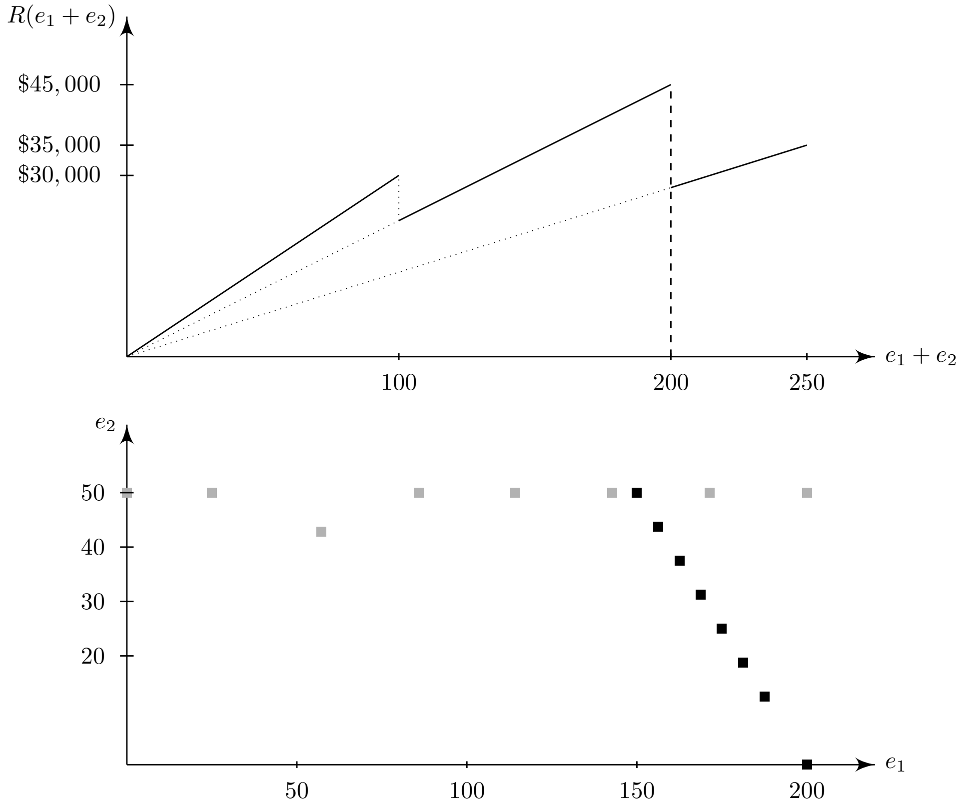

Figure 2: No Nash equilibria

In our problem, by allowing the price makers to bid any amount within their range and, thus, creating a continuous strategy space, we conjecture the existence of at least one Nash equilibrium.

***Conjecture* 1.** *The bidding problem* $\Gamma_t[\cdot]$ *for hydroelectric producers with payoff function* (3)-(7)*, always yields at least one Nash equilibrium. Further, the cooperative solution,* i.e., *the point that maximizes joint revenue, is always a Nash equilibrium.*

From an economic point of view, it is very uncommon for the collaborative and the competitive solutions to be the same. Economists typically welcome competitive solutions, but study policy instruments that aim to prevent collusion. In our problem, the collaborative solution is a competitive solution and this occurs because of the non-concave revenue function. It is important to note that if the price makers collectively act as a single firm they can still influence prices in their favor and exercise market power (see [21]). The existence of multiple Nash equilibria and their location depends on the shape of the revenue function. Consider the following three cases:

(i.) Joint revenue is maximized when all price makers bid their capacity. In this case, at least one Nash equilibrium exists where all price makers produce at capacity and this equilibrium is Pareto optimal (see Figure 8).

(ii.) Joint revenue is maximized at some point less than the cumulative price makers' capacity. In this case, the point that maximizes the joint revenue is at a break point, call it E^*. There may be an infinite number of Nash equilibria at which $\sum_n e_n = E^*$ and possibly another Nash equilibrium at the point at which all price-maker producers are producing at capacity (see Figure 5).

(iii.) Joint revenue is maximized when all price makers bid their capacity and at some point less than the cumulative price makers' capacity (*i.e.,* we have alternate optima for the joint revenue). In this case, there are exactly two Nash equilibria (see Figure 6).

The cumulative amount, e_t, at which joint revenue is maximized can be computed, for instance, with the methodology proposed in Steeger [45]. However, it is non-trivial to determine how much energy each individual producer should bid to supply the cumulative amount. In our work below, we utilize a different approach and compute a Nash equilibrium directly.

4.3. Feasibility Problem and Best Response Formulations

To find a Nash equilibrium using optimization, if it exists, practitioners solve the feasibility problem (FP). For our problem and $n = 1, \dots, N$ players,

$$
\begin{aligned}
(FP) \qquad & \max\ z = 1 & & \\
\text{s.t.}\ \ & R_{nt}(e_{nt}, e_{-nt}) + \beta_{n,t+1}(\mathbf{v}_{n,t+1}, \mathbf{v}_{-n,t+1}) \geq \gamma_{nt}(e_{-nt}, \mathbf{V}_{nt}, \mathbf{v}_{-n,t+1}) & \forall\, n,t \quad & (8)\\
& v_{i,n,t+1} = V_{int} + A_{int} - \frac{e_{int}}{\rho_{in}} - s_{int} + \sum_{m \in \mathbb{U}_{in}} \left(\frac{e_{mnt}}{\rho_{mn}} + s_{mnt} \right) & \forall\, i,n,t \quad & (9)\\
& \underline{V}_{in} \leq v_{i,n,t+1} \leq \overline{V}_{in} & \forall\, i,n,t \quad & (10)\\
& 0 \leq \frac{e_{int}}{\rho_{in}} \leq \overline{U}_{in} & \forall\, i,n,t \quad & (11)\\
& e_{nt} = \sum_{i \in \mathbb{I}_n} e_{int} & \forall\, n,t \quad & (12)\\
& s_{int} \geq 0 & \forall\, i,n,t. \quad & (13)
\end{aligned}
$$

In (FP), the $R_{nt}(\cdot)$ functions are found via (4)-(7). Constraint (8) ensures that every price maker's production level is a best response to all other price makers' production levels. This is equivalent to (2). The right-hand side $\gamma_{nt}(\cdot)$ functions, or best response functions, are computed a priori for each of the price makers assuming the bids and water levels from all other price makers are known. We discuss this in greater detail in the following paragraph. Interpolations of the $\gamma_{nt}(\cdot)$ functions are created so that we have right-hand side values for any of the function's parameters. Constraint (9) establishes the conditions necessary to connect each of the stages in the time horizon, via the state variables $v_{i,n,t+1}$. Last, (10) -(13) define the bounds on each of the decision variables.

The function $\gamma_{nt}(E_{-nt}, \mathbf{V}_{nt}, \mathbf{V}_{\omega,t+1})$ is price maker n's maximum achievable revenue, given bids from all other price makers (E_{-nt}). Thus, if (8) holds, then none of the price makers can increase their revenue, given all other price makers' bids, and this defines a Nash equilibrium. To obtain $\gamma_{nt}(E_{-nt}, \mathbf{V}_{nt}, \mathbf{V}_{-n,t+1})$, for any n, and create their interpolations, we solve (BR_n). In (BR_n) we assume we know both the bids and the reservoir levels for all other price makers E_{-nt} and $\mathbf{V}_{-n,t+1}$.

$$
\begin{aligned}
(BR_n) \qquad \gamma_{nt}(E_{-nt}, \mathbf{V}_{nt}, \mathbf{V}_{-n,t+1}) := \max\ & R_{nt}(e_{nt}, E_{-nt}) + \beta_{n,t+1}(\mathbf{v}_{n,t+1}, \mathbf{V}_{-n,t+1}) \\
\text{s.t.}\ & e_{nt} + \sum_{-n} E_{-nt} \le E'_{kt} + \mathcal{M}\phi_{kt} && \forall\, k \\
& R_{nt}(e_{nt}, E_{-nt}) \le \pi^d_{kt} e_{nt} + \mathcal{M}\phi_{kt} && \forall\, k \\
& v_{i,n,t+1} = V_{int} + A_{int} - e_{int}/\rho_{in} - s_{int} \\
& \quad + \sum_{m \in \mathbb{U}_{in}} \left(\frac{e_{mnt}}{\rho_{mn}} + s_{mnt} \right) && \forall\, i \\
& \underline{V}_{in} \le v_{i,n,t+1} \le \overline{V}_{in} && \forall\, i \\
& 0 \le \frac{e_{int}}{\rho_{in}} \le \overline{U}_{in} && \forall\, i \\
& e_{nt} = \sum_{i=1}^{I_n} e_{int} \\
& \sum_{k=1}^{K} (1 - \phi_{kt}) = 1 \\
& \phi_{kt} \in \{0, 1\} && \forall\, k \\
& s_{int} \ge 0 && \forall\, i.
\end{aligned}
$$

When seeking an equilibrium, most practitioners utilize an iterative approach that, for all intents and purposes, solves (FP). If multiple Nash equilibria exist, the approach will not provide any indication. Additionally, this approach does not guarantee a Pareto-optimal equilibrium will be selected, if one exists. All that the approach guarantees is that the solution will be a Nash equilibrium. For these reasons, we propose a different approach.

4.4. Solution Approach

For ease of presentation, we assume there are only two price makers. In this approach, the function $\gamma_{1t}(\cdot)$ is solved for multiple, incremental, and discrete values of E_{2t}, $\mathbf{V}_{1t}$, and $\mathbf{V}_{2,t+1}$, *i.e.*, $(E_{2t}, \mathbf{V}_{1t}, \mathbf{V}_{2,t+1}) \in \mathbb{D}_1$ (see Algorithm 1). The same is done for $\gamma_{2t}(\cdot)$. We then create an interpolation of the $\gamma(\cdot)$ functions using bilinear interpolation. These interpolations are denoted by $\tilde{\gamma}(\cdot)$ and are used in (8).

Adding a recursive objective function and constraints (4)-(7) with $\omega = 1$ to (FP) yields the strategic bidding problem for price maker one

$$
\begin{aligned}
(SB_1) \qquad \beta_{1t}(\mathbf{V}_{1t}, \mathbf{V}_{2t}) := \max\ & \psi + \beta_{1,t+1}(\mathbf{v}_{1,t+1}, \mathbf{v}_{2,t+1}) \\
\text{s.t. }\ & (4) - (7) \text{ with } \omega = 1, (8) - (13) \\
& R_{2t}(e_{1t}, e_{2t}) \leq \pi^d_{kt} e_{2t} + \mathcal{M}\phi_{kt} \qquad \forall\ k \\
& R_{2t}(e_{1t}, e_{2t}) \geq \pi^d_{kt} e_{2t} - \mathcal{M}\phi_{kt} \qquad \forall\ k.
\end{aligned}
$$

We can be certain that (SB_1) yields a Nash equilibrium, when it exists, because the constraint set includes all of the constraints in (FP).

Similarly, for price maker two with $\omega = 2$, we have

$$
\begin{aligned}
(SB_2) \qquad \beta_{2t}(\mathbf{V}_{1t}, \mathbf{V}_{2t}) := \max\ & \psi + \beta_{2,t+1}(\mathbf{v}_{1,t+1}, \mathbf{v}_{2,t+1}) \\
\text{s.t. }\ & (4) - (7) \text{ with } \omega = 2, (8) - (13) \\
& R_{1t}(e_{1t}, e_{2t}) \leq \pi^d_{kt} e_{1t} + \mathcal{M}\phi_{kt} \qquad \forall\ k \\
& R_{1t}(e_{1t}, e_{2t}) \geq \pi^d_{kt} e_{1t} - \mathcal{M}\phi_{kt} \qquad \forall\ k.
\end{aligned}
$$

Solving (SB_1) or (SB_2) results in optimal hydro production $e^*_{nt}\ \forall\ n$ and optimal reservoir volumes $\mathbf{v}^*_{n,t+1}\ \forall\ n$. With these optimal quantities, we define

$$
\beta_{nt}(\mathbf{V}_{nt}, \mathbf{V}_{-nt}) := R_{nt}(e^*_{nt}, e^*_{-nt}) + \beta_{n,t+1}(\mathbf{v}^*_{n,t+1}, \mathbf{v}^*_{-n,t+1}) \qquad \forall\ n,\ -n.
$$

Once the $\tilde{\gamma}(\cdot)$ functions are populated for stage t we solve (SB_1) or (SB_2) for stage t. We continue this procedure recursively from $t = T$ to $t = 1$. The backwards recursion yields the desired results because with $\beta_{1,T+1}(\cdot) = 0$ and $\beta_{2,T+1}(\cdot) = 0$, $\beta_{1T}(\cdot) = R^*_{1T}(\cdot)$ and $\beta_{2T}(\cdot) = R^*_{2T}(\cdot)$, and thus

$$
\begin{aligned}
&\beta_{1,T-1}(\cdot) = R^*_{1,T-1}(\cdot) + \beta_{1T}(\cdot) = R^*_{1,T-1}(\cdot) + R^*_{1T}(\cdot), \text{ and} \\
&\beta_{2,T-1}(\cdot) = R^*_{2,T-1}(\cdot) + \beta_{2T}(\cdot) = R^*_{2,T-1}(\cdot) + R^*_{2T}(\cdot),
\end{aligned}
$$

and so forth. This continues until we have $\beta_{11}(\cdot)$ and $\beta_{21}(\cdot)$.

Exactly how these formulations are used in conjunction with our dynamic programming algorithm is shown in Algorithm 1. Though this approach yields a Nash equilibrium, if multiple equilibria exist, (SB_1) and (SB_2) may not yield the same Nash equilibrium as each other or the one found in (FP). In (SB_1), we seek a Nash equilibrium that maximizes price maker one's revenue and vice versa for (SB_2). Consequently, price make one has an advantage in (SB_1) and price maker two has an advantage in (SB_2). We discuss how we resolve this issue in the following section.

4.5. Multiple Nash Equilibria

When it exists, the solutions to (SB_1) and (SB_2) yield a Nash equilibrium that is Pareto optimal. What this means is that if there are multiple equilibria, then one equilibrium will be preferred by

Algorithm 1 DP algorithm for two price makers ($t = 1, \ldots, T$).

Step (0). Initialize: $\beta_{1,T+1}(\mathbf{V}_{1,T+1}, \mathbf{V}_{2,T+1}) = 0$
for $t = T, T-1, \ldots, 1$ **do**
  for $(E_{2t}, \mathbf{V}_{1t}, \mathbf{V}_{2,t+1}) \in \mathbb{D}_1$ **do**
    Step (1). Solve (BR_1) for $\gamma_{1t}(E_{2t}, \mathbf{V}_{1t}, \mathbf{V}_{2,t+1})$ using the interpolation $\tilde{\beta}_{1,t+1}(\mathbf{v}_{1,t+1}, \mathbf{V}_{2,t+1})$.
  end for
  for $(E_{1t}, \mathbf{V}_{1,t+1}, \mathbf{V}_{2t}) \in \mathbb{D}_2$ **do**
    Step (2). Solve (BR_2) for $\gamma_{2t}(E_{1t}, \mathbf{V}_{1,t+1}, \mathbf{V}_{2t})$ using the interpolation $\tilde{\beta}_{2,t+1}(\mathbf{V}_{1,t+1}, \mathbf{v}_{2,t+1})$.
  end for
  Step (3). Construct the interpolations $\tilde{\gamma}_{1t}(\cdot)$ and $\tilde{\gamma}_{2t}(\cdot)$.
  for $(\mathbf{V}_{1t}, \mathbf{V}_{2t}) \in \mathbb{D}_3$ **do**
    Step (4). Solve (SB_1) for stage t using $\tilde{\gamma}_{2t}(\cdot)$.
    Step (5). Solve (SB_2) for stage t using $\tilde{\gamma}_{1t}(\cdot)$.
    Step (6). Construct the interpolations $\tilde{\beta}_{1t}(\mathbf{V}_{1,t+1}, \mathbf{V}_{2,t+1})$ and $\tilde{\beta}_{2t}(\mathbf{V}_{1,t+1}, \mathbf{V}_{2,t+1})$.
  end for
end for
return $\gamma_{1t}(E_{2t}, \mathbf{V}_{1t}, \mathbf{V}_{2,t+1})$ and $\gamma_{2t}(E_{1t}, \mathbf{V}_{1,t+1}, \mathbf{V}_{2t})$ for discrete values of E_{2t}, $\mathbf{V}_{1t}$, $\mathbf{V}_{2,t+1}$, E_{1t}, $\mathbf{V}_{1,t+1}$, and $\mathbf{V}_{2t}$ $\forall\ t$; and $\beta_{1t}(\mathbf{V}_{1t}, \mathbf{V}_{2t})$ and $\beta_{2t}(\mathbf{V}_{1t}, \mathbf{V}_{2t})$, e^*_{1t}, e^*_{2t}, $\mathbf{v}^*_{1,t+1}$, and $\mathbf{v}^*_{2,t+1}$ $\forall\ t$.

both price makers to all of the other equilibria and the solution to both (SB_1) and (SB_2) will be this point [35]. In the absence of an equilibrium that is Pareto optimal, (SB_1) and (SB_2) will yield different equilibria, and these equilibria are Pareto efficient. In other words, if there are multiple equilibria and none of them are strictly and jointly preferred to the others, by both price makers, then (SB_1) and (SB_2) will yield two different solutions. In both of these solutions, any change in the production quantities cannot increase one price-maker's revenue without decreasing the other's.

If the solutions to (SB_1) and (SB_2) do not agree, then there are two or more equilibria. If, in the solutions, the market-clearing price and aggregate production is the same, then there is an infinite number of Pareto efficient Nash equilibria that can be represented as any convex combination of the solutions to (SB_1) and (SB_2). Formally stated,

***Theorem* 1.** *If (SB_1) and (SB_2) yield different solutions in which the market-clearing price is the same and the aggregate production is the same, then every convex combination of the solutions to (SB_1) and (SB_2) is a Nash equilibrium.*

The proof of Theorem 1 is given in Appendix A.

Figure 3 shows how a unique solution is determined in all cases. If (SB_1) and (SB_2) yield identical results then we have a Pareto-optimal equilibrium and are done. However, if the results do not agree, we use Nash's solution (see Section 3) to determine the equilibrium to implement. The difficulty with this problem, in this application, is selecting the disagreement payments, R^d_{1t} and R^d_{2t}. We assume that the disagreement payoff for each price-maker is the minimum revenue

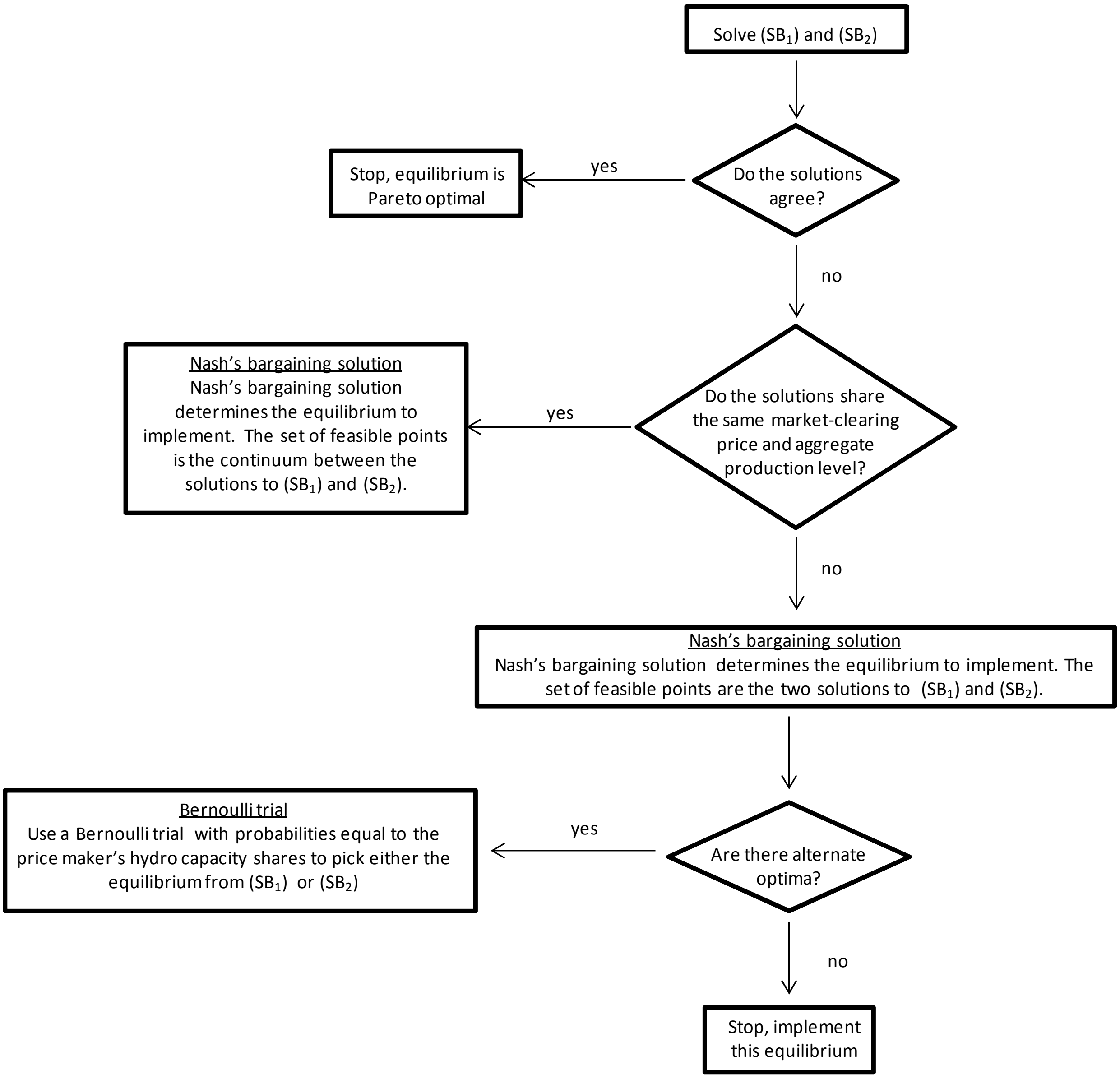

Figure 3: Multiple Nash equilibria flow chart

they receive if they are responding optimally. For price-maker one, this equates to solving

$$
\begin{aligned}
R^d_{1t} := \; & \min \; \psi + \beta_{1,t+1}(\mathbf{v}_{1,t+1}, \mathbf{v}_{2,t+1}) \\
& \text{s.t.} \;\; (4) - (7) \text{ for } \omega = 1, (9) - (13) \\
& \qquad \psi \geq \pi^d_{kt} e_{1t} - \mathcal{M}\phi_{kt} \\
& \qquad \psi + \beta_{1,t+1}(\mathbf{v}_{1,t+1}, \mathbf{v}_{2,t+1}) \geq \gamma_{1t}(e_{2t}, \mathbf{V}_{1t}, \mathbf{v}_{2,t+1}).
\end{aligned}
$$

R^d_{1t} does not have to be a Nash equilibrium because price-maker two is not responding optimally to price-maker one. In other words, R^d_{1t} is not necessarily the revenue-minimizing Nash equilibrium for price maker-one. R^d_{2t} is found in a parallel manner and similar arguments can be made for it.

If Nash's solution yields alternate optima, then we select the solution to (SB_1) with probability ξ_1 and the solution to (SB_2) with probability ξ_2. For price maker ω, $\xi_\omega = \frac{a}{b}$, where

$$a := \sum_i \min\left(v_{i\omega t} + A_{i\omega t} - \underline{V}_{i\omega t} + \sum_{m \in \mathbb{U}_{i\omega}} \left(\frac{e_{m\omega t}}{\rho_{m\omega}} + s_{m\omega t}\right), \overline{U}_{i\omega}\rho_{i\omega}\right)$$

and

$$b := \sum_n \sum_i \min\left(v_{int} + A_{int} - \underline{V}_{int} \sum_{m \in \mathbb{U}_{in}} \left(\frac{e_{mnt}}{\rho_{mn}} + s_{mnt}\right), \overline{U}_{in}\rho_{in}\right).$$

We term this value the price maker's hydro capacity share. In other words, we base each price maker's bargaining power off of the ratio of their hydro capacity to total hydro capacity.

5. Computational Results

Our computational tests are performed on a standard desktop computer with an Intel(R) Core(TM) i3 CPU @ 2.10 GHz processor, with 4GB of RAM, running on Windows 7. We implemented the algorithms in GAMS 24.1.3 and solved the resulting MILPs using CPLEX version 12.5.1.0.

To better illustrate the finer points of our methodology, we present three small examples and a case study modeling Honduras' electricity market. For ease of presentation and comprehension, in each of the examples we solve a static model (*i.e.,* assume $T=1$). Each of the three illustrative examples depicts a different path in Figure 3. Also, in each of the three illustrative examples, we supply enough inflow so that each best response function is only a function of the other price maker's bids and not their own water level. This allows us to plot the best response functions in 2-dimensions.

5.1. Illustrative Example 1

Assume there are two price-taker thermal producers and two price-maker hydro producers (each operating one reservoir) with parameters given in Table 4.

Table 4: Example 1 – Thermal and hydro parameters

j	$\overline{G}_j$	C_j	g_j
1	300	140	300
2	120	225	120
3	150	300	100
D	520		

i	V_{0i}	$\underline{V}_i$	$\overline{V}_i$	ρ_i	$\overline{U}_i$	A_{i1}
$n=1$						
1	0	0	400	1	151	350
$n=2$						
1	0	0	200	1	200	200

The joint revenue and best response functions are plotted in Figure 4. From the plot, we see that there are two Nash equilibria. One is at the point ($e_1 = 95, e_2 = 125$) and the other occurs when both hydro producers produce at capacity, or ($e_1 = 151, e_2 = 200$). Both (SB_1) and (SB_2) yield the solution ($e_1 = 95, e_2 = 125$), because this Nash equilibrium is Pareto optimal. The equilibrium is Pareto optimal because at ($e_1 = 95, e_2 = 125$), $R_1(e_1) = \$21,375$ and $R_2(e_2) = \$28,125$, whereas at ($e_1 = 151, e_2 = 200$), $R_1(e_1) = \$21,140$ and $R_2(e_2) = \$28,000$. In this case, both hydro producers prefer ($e_1 = 95, e_2 = 125$) and our method finds this solution.

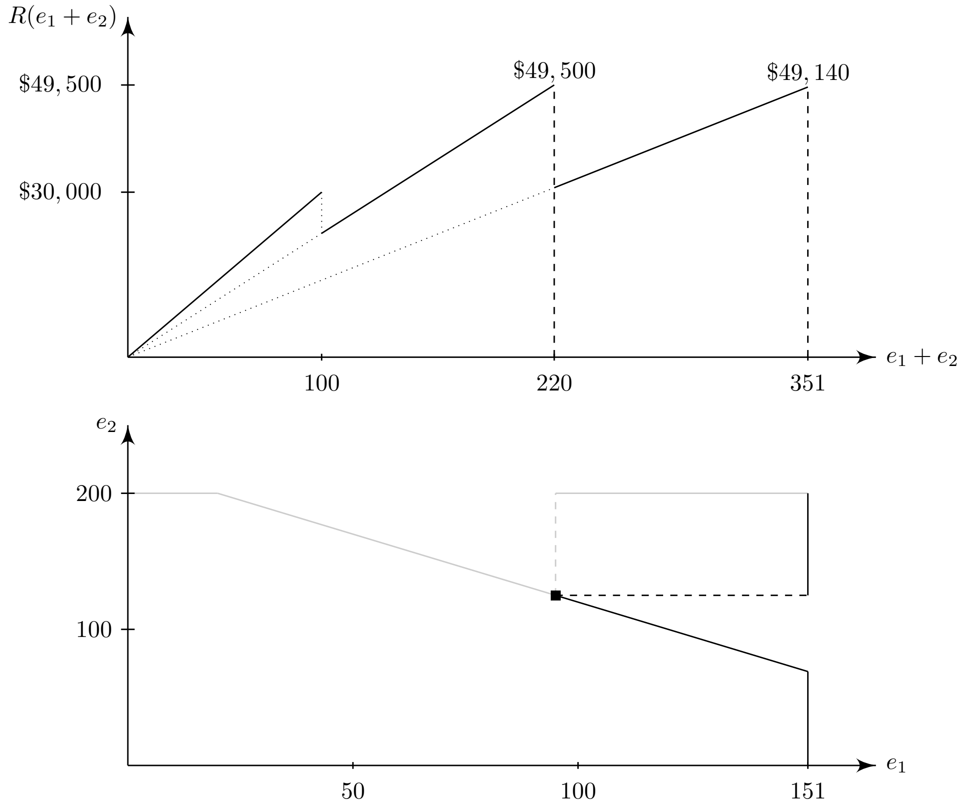

Figure 4: Illustrative example 1: Revenue and best response functions. The Pareto-optimal equilibrium is marked.

5.2. Illustrative Example 2

Again, assume there are two price-taker thermal producers and two price-maker hydro producers, that each operate one reservoir (parameters are given in Table 5).

The joint revenue and best response functions are plotted in Figure 5. From the plot we see that there are an infinite number of Nash equilibria in the range $e_1 \in (150, 168.88)$, $e_2 \in (31.12, 50)$ in which the cumulative production equals 200 GWh's. Solving for a Nash equilibrium using (SB_1) or (SB_2) yields two different solutions (production quantities): ($e_1 = 166.67, e_2 = 33.33$) and ($e_1 = 150, e_2 = 50$). This is expected, because (SB_1) maximizes price maker one's revenue while ensuring the solution is a Nash equilibrium, whereas (SB_2) maximizes price maker two's revenue while ensuring the solution is a Nash equilibrium. To obtain this solution, we approximate the

Table 5: Example 2 – Thermal and hydro parameters

j	$\overline{G}_j$	C_j	g_j
1	300	140	300
2	100	225	100
3	150	300	100
D	500		

i	V_{0i}	$\underline{V}_i$	$\overline{V}_i$	ρ_i	$\overline{U}_i$	A_{i1}
$n = 1$						
1	0	0	400	1	200	350
$n = 2$						
1	0	0	200	1	50	200

gamma functions by partitioning each xy space into thirty-six polygons. Since we are approximating the gamma functions, our solution is not exact. The actual solution is ($e_1 = 168.88, e_2 = 31.12$) and ($e_1 = 150, e_2 = 50$).

In Nash's bargaining solution, we maximize the excess revenue gained from reaching an agreement. Mathematically, we solve

$$\begin{aligned} \max_{e_1,e_2}\ & \left(R_1(e_1,e_2) - R_1^d\right)\left(R_2(e_1,e_2) - R_2^d\right) \\ & = (225e_1 - 33,750)\,(225e_2 - 7,000) \\ \text{s.t. } & e_1 + e_2 \leq 200 \\ & 150.00 \leq e_1 \leq 168.88 \\ & 31.12 \leq e_2 \leq 50.00. \end{aligned}$$

In this formulation, $R_1^d = \$33,750$ and $R_2^d = \$7,000$ are found as discussed in Section 4.5. Since we know the optimal production occurs when $e_1 + e_2 = 200$, we know the slope of each price-maker's revenue function and can represent their revenue accordingly. Typically, we would add the constraints $R_1(e_1) \geq R_1^d$ and $R_2(e_2) \geq R_2^d$, but these are redundant because any revenue for price-maker one and two along the line at which $e_1 + e_2 = 200$, will satisfy these constraints. Solving for Nash's bargaining solution yields $e_1 = 159.44$ and $e_2 = 40.56$.

5.3. Illustrative Example 3

For the third example, we again assume there are two price-taker thermal producers and two price-maker hydro producers (each operating one reservoir); however, the parameters change slightly (see Table 6).

Table 6: Example 3 – Thermal and hydro parameters

j	$\overline{G}_j$	C_j	g_j
1	300	140	300
2	100	245	100
3	150	300	100
D	500		

i	V_{0i}	$\underline{V}_i$	$\overline{V}_i$	ρ_i	$\overline{U}_i$	A_{i1}
$n = 1$						
1	0	0	400	1	150	350
$n = 2$						
1	0	0	200	1	200	200

The joint revenue and best response functions are plotted in Figure 6. Solving (SB_1) yields production quantities $e_1 = 86$ and $e_2 = 114$ with revenues $R_1(e_1) = \$21,070$ and $R_2(e_2) = \$27,930$, whereas solving ($SB_2$) yields production quantities $e_1 = 150$ and $e_2 = 200$ with revenues $R_1(e_1) = \$21,000$ and $R_2(e_2) = \$28,000$. In this case, ($SB_1$) and ($SB_2$) yield solutions that do not share the

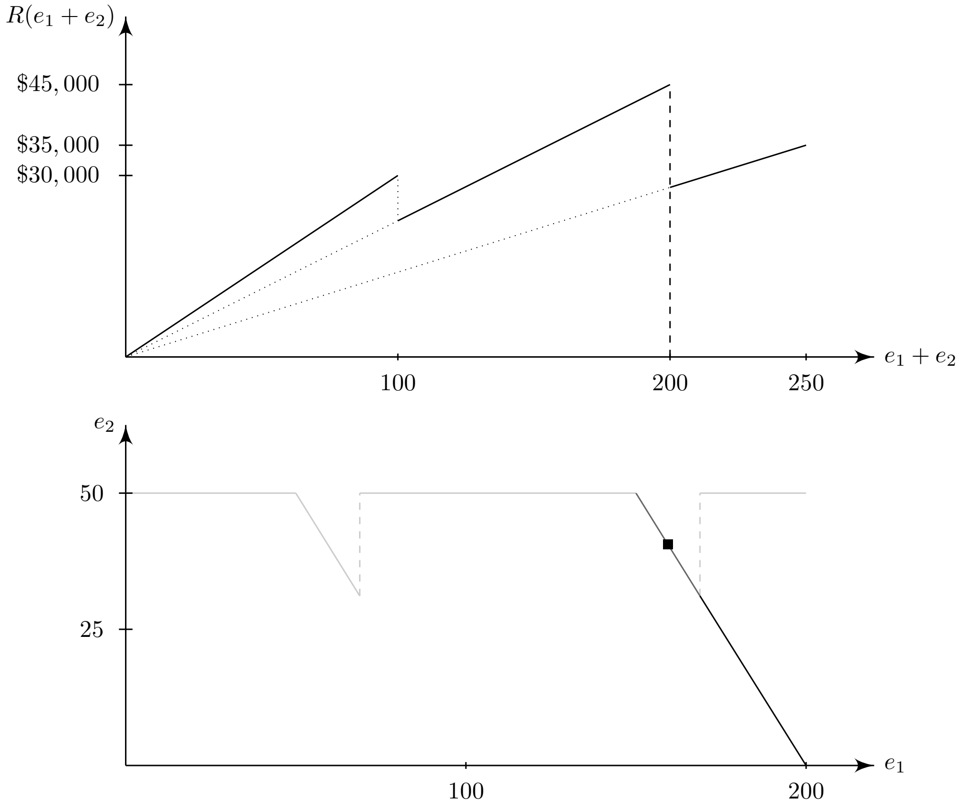

Figure 5: Illustrative example 2: Revenue and best response functions. The bargaining solution is marked.

same market-clearing price. To determine which solution to implement we again use the solution to Nash's bargaining problem, with $R_1^d = \$21,000$ and $R_2^d = \$27,930$. In Nash's solution, both feasible points (equilibria) yield the same objective function value so we have alternate optima. To determine which solution to implement we run a Bernoulli trial in which we select the solution from (SB_1) with probability $\xi_1 = \frac{150}{350} = 42.86\%$ and the solution from (SB_2) with probability (SB_2) with probability $\xi_2 = \frac{200}{350} = 57.14\% = 1 - \xi_1$.

5.4. Case Study

To demonstrate how our method can be used in a real-world example, we model Honduras' electricity market first over a month-long time horizon (one stage) and then over a year-long time horizon (twelve stages). We assume all of the thermal producers, with parameters given in Table 7, act as price takers and that there are two price-maker hydro producers, with parameters given in Figure 7 and Table 8. Additionally, we assume inflows are known with certainty. Run-of-the-river plants are modeled as reservoirs with no storage, *i.e.,* $\underline{V} = \overline{V} = 0$. Initial reservoir levels and inflows are determined based on historical data. Last, if the price-taker thermal producers alone cannot satisfy demand, we assume the price-maker hydro producers must at least produce enough

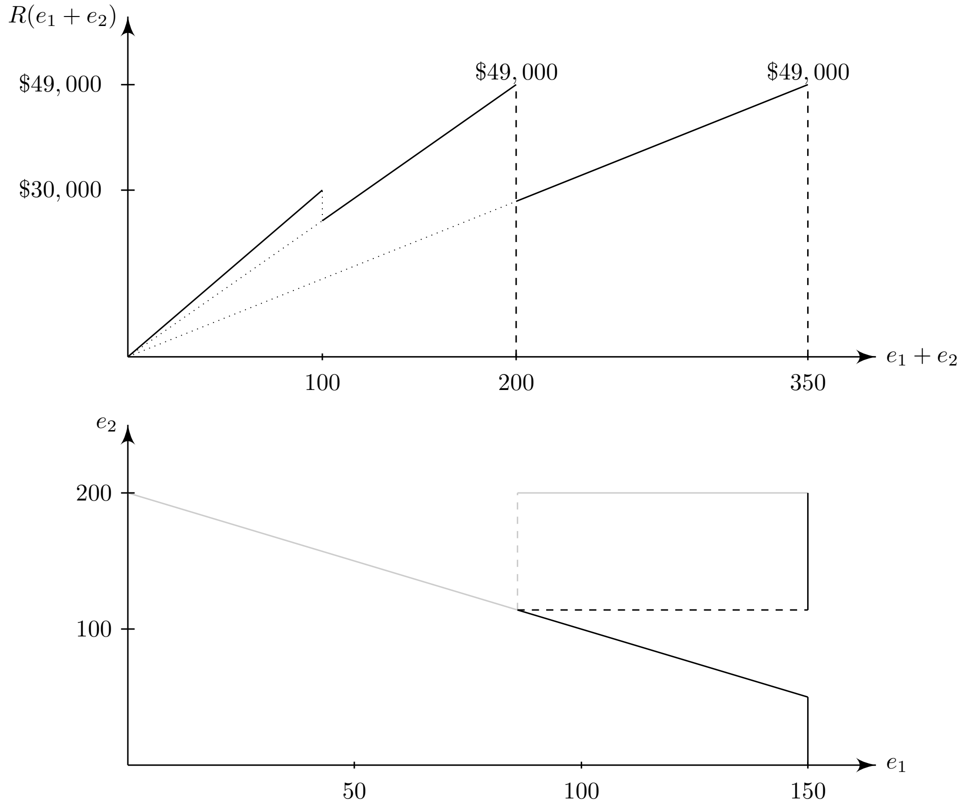

Figure 6: Illustrative example 3: Revenue and best response functions

energy to meet the residual demand. In this case, the price-makers are penalized for not meeting the residual demand and only earn revenue for production levels above the residual demand.

Figure 8 shows the joint revenue and the best response functions for each price maker in the one-stage problem. By limiting the time horizon to one time stage we are able to fix the initial water levels, *i.e.,* $\mathbf{V}_1$ and $\mathbf{V}_2$ are parameters. Consequently, the gamma (or best response) functions are only functions of the other price maker's total production quantities and can be plotted in two dimensions (Figure 9). Solving (SB_1) and (SB_2) for this market yields matching solutions of $\sum_{i\in\mathbb{I}_1} e_i = 236.58$ and $\sum_{i\in\mathbb{I}_2} e_i = 108.78$. From Figure 9, we observe that there is exactly one Nash equilibrium in which both producers produce as much as possible based on the water that is available to them and their turbining capacities. This is precisely the solution that we find through solving (SB_1) and (SB_2). Since we are only modeling the market for one stage, in this instance, the solve times are negligible. In general, solve times increase with the number of divisions used in the interpolations, the number of hydro producers (players), and the number of steps in the time horizon. The number of divisions used in the interpolations have the largest impact.

The above example is relatively "small." We can increase the size of the problem by either modeling a larger market (more thermal generators), by adding more players (hydro producers), or by modeling the problem over more time stages. Adding more thermal generators will only

Table 7: Honduras' thermal parameters

j	**Name**	$\overline{G}_j$	C_j
1	Lufu3-210	156.24	0.075225
2	Emce2	40.92	0.080125
3	Enersa	159.96	0.081225
4	Lufussa2	57.29	0.082375
5	Elcosa	59.52	0.085275
6	Ceiba	17.86	0.091725
7	Lufussa1	29.39	0.162955
8	Puert ENE	11.90	0.253537
9	Puert MEX	7.44	0.299155
	Total	540.52	
	Min	7.44	0.075225
	Max	159.96	0.299155

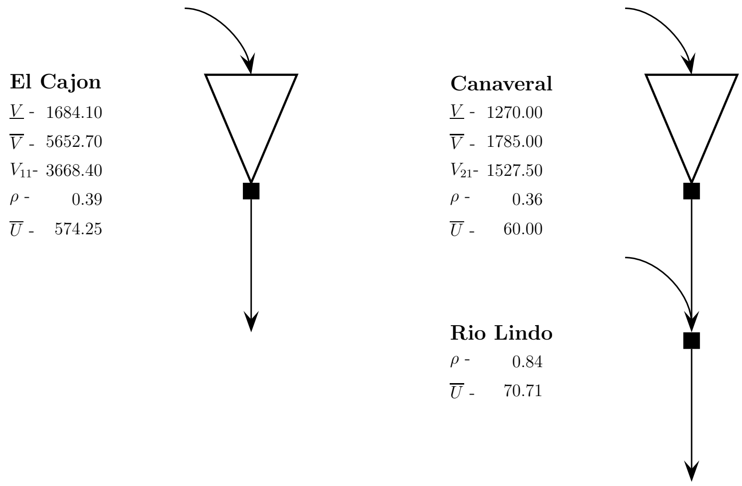

Figure 7: Honduras' hydro profile and parameters. El Cajon is owned and operated by price-maker one, whereas Canaveral and Rio Lindo are owned and operated by price-maker two. Run-of-the-river plants not included in the profile are not connected to reservoirs [45].

affect the shape of the revenue curve and will have no impact on our algorithm or solve times. This is because changes to the revenue curve are addressed in preprocessing steps and then entered as data in the model. As we add more players (hydro producers) we decrease the players' ability to impact the market price and move towards a competitive market or a market in which the players cannot impact prices unless they work together, *i.e.,* the single price-maker problem. That being said, there are cases in which three to four players have market power. To demonstrate how our approach can be used to model larger problem instances we extend the case study and model Honduras' electricity market over a year-long time horizon, with monthly steps.

Again, we assume all of the thermal producers, with parameters given in Table 7, act as price takers and that there are two price-maker hydro producers, with parameters given in Figure 7 and Table 8. Note that the capacities $\overline{G}_j$ and $\overline{U}_i$ will change based on how many days there are in the month being modeled. These changes will alter the shape of the joint revenue curve slightly,

Table 8: Honduras' hydro parameters

n	i	**Name**	ρ_i	$\overline{U}_i$	A_i
1	1	El Cajon	0.39	574.25	381.78
	4	Nispero	0.37	45.26	9.78
	5	Nacaome	0.06	189.10	29.55
	6	Esperanza	1.27	7.50	6.40
2	2	Canaveral	0.36	60.00	42.50
	3	Rio Lindo	0.84	70.71	29.10
	7	Cuyamapa	0.83	9.37	7.99
	8	Cuyamel	0.31	18.75	34.25
	9	Peq-hidro	0.52	29.30	46.98

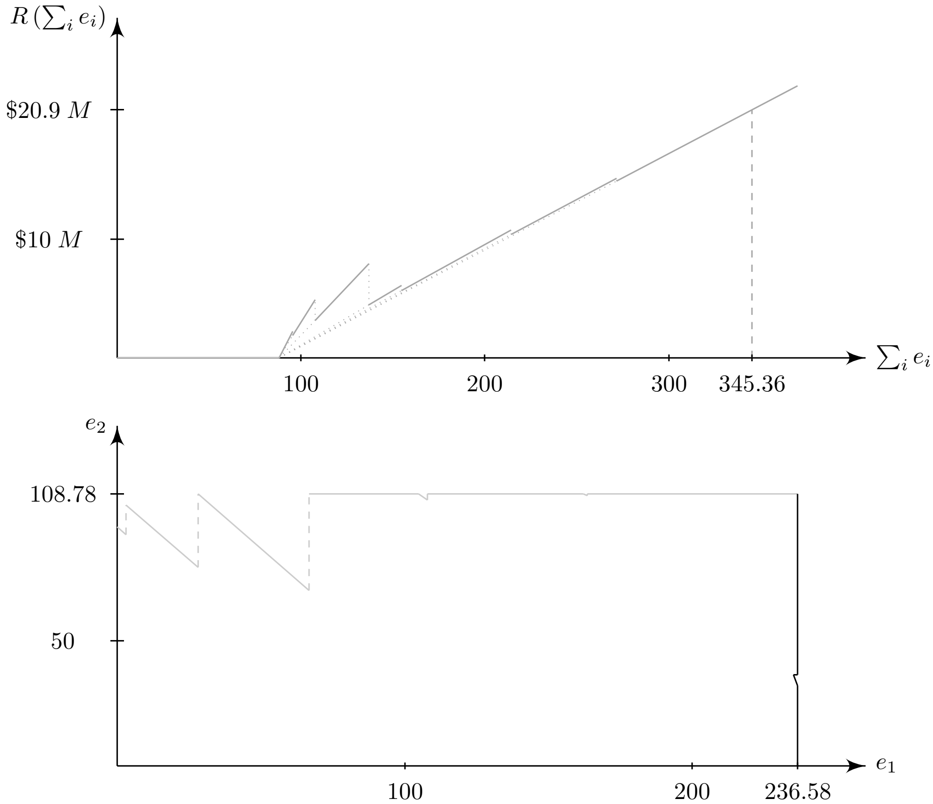

Figure 8: Honduras' revenue and best response functions

however the monthly joint revenue curves will maintain the basic shape given in Figure 8. Figure 10 shows the parameters used and results found from modeling Honduras' electricity market over a year-long time horizon. Hydro inflows are shown in aggregate for all of the hydro plants. Since each price maker owns and operates exactly one reservoir, the initial water levels for each price maker correspond to initial water levels for each reservoir. The initial water levels can also be thought of as storage levels since the initial levels for month *t* are the storage levels for month

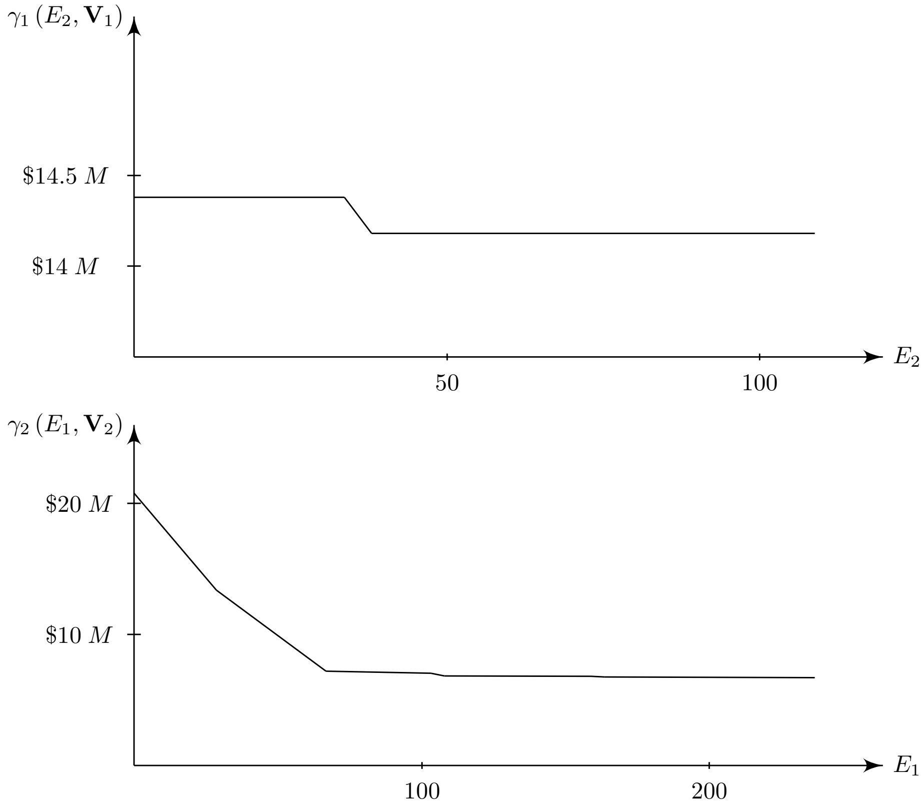

Figure 9: Best response functions

$t - 1$. From the plot we see that the two price makers always produce enough to exceed residual demand and that the production moves in tandem with the demand. In each month, the solutions to both (SB_1) and (SB_2) agree so we can be certain that the solution in each month is Pareto optimal. We use 400 divisions in each interpolation and the computation time for this year-long example is just over 7 minutes.

6. Future Work and Extensions

In order to provide a thorough analysis, we limited our study to one time stage with two price makers and deterministic inflows. Consequently, natural extensions to our methodology could be made by

- studying the problem over longer time horizons;
- including more price maker agents;
- incorporating stochastic inflows (see Remark 6 in Section 2); and/or

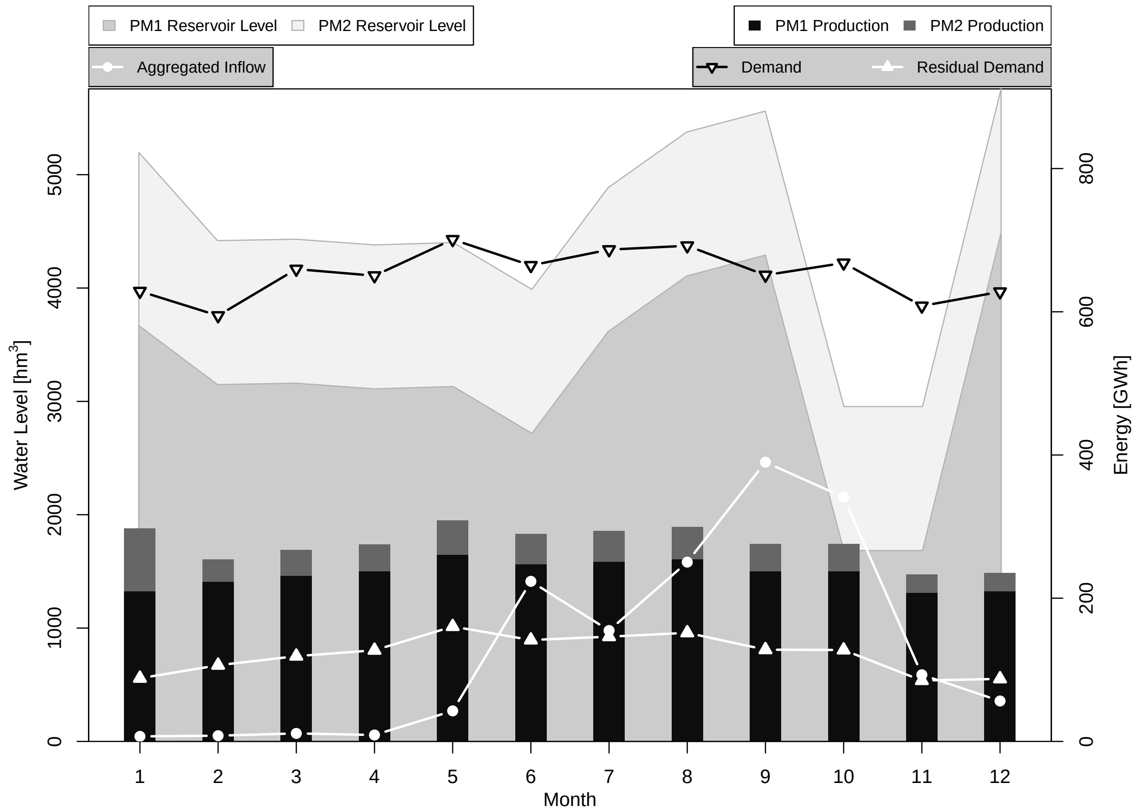

Figure 10: Honduras' electricity market over a year

- improving the interpolations used for the best response functions.

The first two extensions in the above list are not overly complex. Incorporating stochastic inflows is a bit more involved. Since the Nash equilibrium in each stage is dependent on each price maker's production capacity and since each price maker's production capacity is directly linked to inflows, uncertainty in inflows will have a significant impact on the solution. The focus of this paper is not on creating the most efficient and/or most accurate approximations for the best response functions. As such, the approximations could be improved through utilizing techniques likes those in Rebennack and Kallrath [40] or Vielma and Nemhauser [47].

In this particular application, there are four additional areas which future research can focus on:

- mixed strategy equilibria (see Remark 5 in Section 2),
- incorporating DC transmission constraints in the market-clearing formulation [21, 14],
- proving Conjecture 1 (see Remark 1 in Section 2), and
- multi-stage disagreement payoffs in Nash's solution to the bargaining problem.

Throughout this paper we seek a pure strategy equilibrium, *i.e.,* an equilibrium in which each player's strategy is selected with certainty. If we relax this assumption and allow the players to select their strategy based on a probability distribution, then the game changes and we seek a mixed-strategy equilibrium in each stage. Exactly what this means and how it changes our results could be studied in detail. Last, to determine a price-maker's disagreement payoff in a given stage, we examined the problem in a static environment. In other words, we assumed that if the players could not reach an agreement, in a given stage, it would only impact the decision made in that stage. If this assumption is relaxed, then each player's bargaining power could change substantially.

7. Conclusion

This paper presents a method for determining the revenue-maximizing production schedule for multiple price-maker hydro producers. In every stage of the problem, and for every price maker, we seek a production quantity that maximizes their revenue while ensuring all other price-maker producers respond optimally to their revenue-maximizing production quantity. Since the price-maker producers submit quantity bids, the solution, in every stage, is a Nash-Cournot equilibrium. We build interpolations for the best response functions by solving MILPs that are based on discrete functional parameters. To ensure all other price makers respond optimally to a given price maker's revenue-maximizing production quantity, we include the interpolated best response functions in the constraint set of a price maker's MILP.

Typically, this problem is solved via an iterative approach or via a feasibility problem. The limitation with these approaches is that they cannot detect or distinguish between multiple equilibria. This means that a Pareto-optimal equilibrium may be overlooked. Practitioners typically are satisfied when a single Nash equilibrium is found, even in cases in which multiple (or even infinite) Nash equilibria exist. If one exists, our method will select the equilibrium that is Pareto optimal. Iteration can be time consuming and tedious. Our method finds a Nash equilibrium in a single-shot optimization procedure. We demonstrate the usefulness of our approach by presenting several illustrative examples and a case study.

Acknowledgements

We thank the economists Edward Balistreri, Harrison Fell (both Colorado School of Mines) and Daniel Kaffine (University of Colorado, Boulder) for their discussions on game theory.

Appendix A. Proofs

***Theorem* 1.** *If (SB_1) and (SB_2) yield different solutions in which the market-clearing price is the same and the aggregate production is the same, then every convex combination of the solutions to (SB_1) and (SB_2) is a Nash equilibrium.*

Proof. Assume (SB_1) yields solution (e_1^1, e_2^1) with market-clearing price π^d and (SB_2) yields solution (e_1^2, e_2^2) with market-clearing price π^d, at which $e_1^1 + e_2^1 = e_1^2 + e_2^2 = E'$. Observe that E' must equal a breakpoint in the joint revenue function. Without loss of generality, assume $e_1^1 < e_1^2$

and define $0 < \epsilon < e_1^2 - e_1^1$. By contradiction, we show that price-maker two's best response e_2^* to price maker one's bid $e_1^1 + \epsilon$ is $e_2^1 - \epsilon$. Consider

Case $e_2^* < e_2^1 - \epsilon$**:** This contradicts that (e_1^1, e_2^1) is a Nash equilibrium, because price maker two's best response to e_1^1 would be $e_2^* + \epsilon < e_2^1$.

Case $e_2^* > e_2^1 - \epsilon$**:** This contradicts that (e_1^2, e_2^2) is a Nash equilibrium, because $e_2^* + e_1^1 + \epsilon > E'$ with market-clearing price $< \pi^d$ leading to $e_2^* - (e_1^2 - e_1^1 - \epsilon) > e_2^2$ being a best response to e_1^2.

Price maker one's best response to price maker two's bid $e_2^1 - \epsilon$ is $e_1^1 + \epsilon$, which follows from the same logic as above. Thus, the pair of bids $(e_1^1 + \epsilon, e_2^1 - \epsilon)$ defines a Nash equilibrium for any $0 \leq \epsilon \leq e_1^2 - e_1^1$. □

References

[1] J. Barquin, J. Garcia-González, J. Ubeda. 2000. Water value in competitive markets: Dynamic Programming and game theory. *Sixth International Conference on Probabilistic Models Applied to Power Systems* .

[2] L. Barroso, R. D. Carneiro, S. Granville, M. V. Pereira, M. H. Fampa. 2006. Nash equilibrium in strategic bidding: a binary expansion approach. *IEEE Transactions on Power Systems* **21**(2) 629–638.

[3] C. Baslis, A. Bakirtzis. 2011. Mid-term stochastic scheduling of a price-maker hydro producer with pumped storage. *IEEE Transactions on Power Systems* **26**(4) 1856–1865.

[4] S. Borenstein, J. Bushnell, C. R. Knittel. 1999. Market power in electricity markets: Beyond concentration measures. *Energy Journal* **20**(4) 65–88.

[5] R. Branzei, L. Mallozzi, S. Tijs. 2003. Supermodular games and potential games. *Journal of Mathematical Economics* **39**(1) 39–49.

[6] J. Bushnell. 2003. A mixed complementarity model of hydrothermal electricity competition in the western United States. *Operations Research* **51**(1) 80–93.

[7] R. W. Cottle, J.-S. Pang, R. E. Stone. 2009. *The linear complementarity problem*, vol. 60. Siam.

[8] P. Dasgupta, E. Maskin. 1986. The existence of equilibrium in discontinuous economic games I: Theory. *The Review of Economic Studies* **53**(1) 1–26.

[9] S. de la Torre, J. M. Arroyo, A. J. Conejo, J. Contreras, A. Functions. 2002. Price maker self-scheduling in a pool-based electricity market: A mixed-integer LP approach. *IEEE Transactions on Power Systems* **17**(4) 1037–1042.

[10] S. de la Torre, J. Contreras, A. J. Conejo. 2004. Finding multiperiod Nash equilibria in pool-based electricity markets. *IEEE Transactions on Power Systems* **19**(1) 643–651.

[11] G. Debreu. 1952. A social equilibrium existence theorem. *Proceedings of the National Academy of Sciences of the United States of America* **38**(10) 886.

[12] K. Fan. 1952. Fixed-point and minimax theorems in locally convex topological linear spaces. *Proceedings of the National Academy of Sciences* **38**(2) 121.

[13] S.-E. Fleten, T. K. Kristoffersen. 2007. Stochastic programming for optimizing bidding strategies of a nordic hydropower producer. *European Journal of Operational Research* **181**(2) 916–928.

[14] S. Frank, S. Rebennack. 2012. A primer on optimal power flow: Theory, formulation, and practical examples. Tech. rep., Colorado School of Mines. Division of Economics and Business Working Paper 2012-14.

[15] D. Fudenberg, J. Tirole. 1991. *Game theory*. MIT Press, Cambridge, Mass.

[16] A. Garcia, J. D. Reitzes, E. Stacchetti. 2001. Strategic pricing when electricity is storable. *Journal of Regulatory Economics* **20**(3) 223–247.

[17] I. L. Glicksberg. 1951. A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points. *Proceedings of the National Academy of Sciences* **38** 170–174.

[18] G. Gross, D. Finlay. 2000. Generation supply bidding in perfectly competitive electricity markets. *Computational and Mathematical Organization Theory* **6**(1) 83–98.

[19] E. Hasan, F. D. Galiana. 2008. Electricity markets cleared by merit order – part II: Strategic offers and market power. *IEEE Transactions on Power Systems* **23**(2) 372–379.
[20] E. Hasan, F. D. Galiana, A. J. Conejo. 2008. Electricity markets cleared by merit order – part I: Finding the market outcomes supported by pure strategy Nash equilibria. *IEEE Transactions on Power Systems* **23**(2) 361–371.
[21] B. F. Hobbs. 2001. Linear complementarity models of Nash-Cournot competition in bilateral and POOLCO power markets. *IEEE Power Engineering Review* **21**(5) 63.
[22] B. F. Hobbs, U. Helman. 2004. *Modeling Prices in Competitive Electricity Markets: Chapter 3 Complementarity-Based Equilibrium Modeling for Electric Power Markets*. New York: Wiley.
[23] B. Hobbs, C. Metzler, J. Pang. 2000. Strategic gaming analysis for electric power systems: An MPEC approach. *IEEE Transactions on Power Systems* **15**(2) 638–645.
[24] A. Kannan, U. V. Shanbhag, H. M. Kim. 2011. Strategic behavior in power markets under uncertainty. *Energy Systems* **2** 115–141.
[25] R. Kelman, L. A. Barroso, M. V. F. Pereira. 2001. Market power assessment and mitigation in hydrothermal systems. *IEEE Transactions on Power Systems* **16**(3) 354–359.
[26] D. Kostamis, I. Duenyas. 2009. Quantity commitment, production and subcontracting with bargaining. *IIE Transactions* **41**(8) 677–686.
[27] N. S. Kukushkin. 2011. Nash equilibrium in compact-continuous games with a potential. *International Journal of Game Theory* **40**(2) 387–392.
[28] S. Laengle, G. Loyola. 2012. Bargaining and negative externalities. *Optimization Letters* **6**(3) 421–430.
[29] L. Mallozzi. 2011. An application of optimization theory to the study of equilibria. *EUROGEN*.
[30] A. Mas-Colell, M. D. Whinston, J. R. Green. 1995. *Microeconomic Theory*. Oxford University Press, USA.
[31] V. Nanduri, T. K. Das. 2009. A reinforcement learning algorithm for obtaining the Nash equilibrium of multi-player matrix games. *IIE Transactions* **41**(2) 158–167.
[32] J. F. Nash Jr. 1950. The bargaining problem. *Econometrica: Journal of the Econometric Society* 155–162.
[33] J. F. Nash Jr. 1951. Non-cooperative games. *The Annals of Mathematics* **54**(2) 286–295.
[34] B. Panicucci, M. Pappalardo, M. Passacantando. 2009. On solving generalized Nash equilibrium problems via optimization. *Optimization Letters* **3**(3) 419–435.
[35] P. Pardalos, A. Migdalas, L. Pitsoulis. 2008. *Pareto optimality, game theory and equilibria*, vol. 17. Springer.
[36] B. Philippe. 2009. Existence of pure Nash equilibria in discontinuous and non quasiconcave games. *International Journal of Game Theory* **38**(3) 395–410.
[37] H. M. I. Pousinho, J. Contreras, A. G. Bakirtzis, J. Catalao. 2013. Risk-constrained scheduling and offering strategies of a price-maker hydro producer under uncertainty. *IEEE Transactions on Power Systems* **28**(2) 1879–1887.
[38] D. Pozo, J. Contreras. 2011. Finding multiple Nash equilibria in pool-based markets: A stochastic EPEC approach. *IEEE Transactions on Power Systems* **26**(3) 1744–1752.
[39] A. Ramos, M. Ventosa, M. Rivier. 1999. Modeling competition in electric energy markets by equilibrium constraints. *Utilities Policy* **7**(4) 233–242.
[40] S. Rebennack, J. Kallrath. 2014. Continuous Piecewise Linear Delta-Approximations for Univariate Functions: Computing Minimal Breakpoint Systems. *Journal of Optimization Theory and Applications* doi: 10.1007/s10957-014-0687-3.
[41] P. J. Reny. 1999. On the existence of pure and mixed strategy Nash equilibria in discontinuous games. *Econometrica* **67**(5) 1029–1056.
[42] J. Roberts, H. Sonnenschein. 1976. On the existence of Cournot equilbrium without concave profit functions. *Journal of Economic Theory* **13**(1) 112–117.
[43] T. Scott, E. Read. 1996. Modelling hydro reservoir operation in a deregulated electricity market. *International Transactions in Operational Research* **3**(34) 243–253.
[44] M. Sion, P. Wolfe. 1957. On a game without a value. *In Contributions to the theory of games (Princeton Annals of Mathematical Studies)* **3** 299–306.
[45] G. Steeger. 2014. Strategic bidding for price-maker hydroeletric producers. Ph.D. thesis.
[46] G. Steeger, L. A. Barroso, S. Rebennack. 2014. Optimal bidding strategies for hydro-electric producers: A

literature survey. *IEEE Transactions on Power Systems* **29**(4) 1758–1766.

[47] J. P. Vielma, G. L. Nemhauser. 2011. Modeling disjunctive constraints with a logarithmic number of binary variables and constraints. *Mathematical Programming* **128**(1) 49–72.

[48] J. Villar, H. Rudnick. 2003. Hydrothermal market simulator using game theory: assessment of market power. *IEEE Transactions on Power Systems* **18**(1) 91–98.

[49] W. Xian, L. Yuzeng, Z. Shaohua. 2004. Oligopolistic equilibrium analysis for electricity markets: a nonlinear complementarity approach. *IEEE Transactions on Power Systems* **19**(3) 1348–1355.

[50] Y. Yang, Y. Zhang, F. Li, H. Chen. 2012. Computing all Nash equilibria of multiplayer games in electricity markets by solving polynomial equations. *IEEE Transactions on Power Systems* **27**(1) 81–91.