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Abstract

Direct current (DC) electricity distribution systems have been proposed as an alternative to traditional, alternating current (AC) distribution systems for commercial buildings. Partial replacement of AC distribution with DC distribution can improve service to DC loads and overall building energy efficiency. This article develops (i) a mixed-integer, nonlinear, nonconvex mathematical programming problem to determine maximally energy efficient designs for mixed AC-DC electricity distribution systems in commercial buildings, and (ii) describes a tailored global optimization algorithm based on Nonconvex Generalized Benders Decomposition. The results of three case studies demonstrate the strength of the decomposition approach compared to state-of-the-art general-purpose global solvers.

Keywords: Electric power systems, mixed AC-DC electricity distribution, global optimization, mixed-integer nonlinear programming (MINLP), Nonconvex Generalized Benders Decomposition (NGBD), optimal power flow (OPF)

1. Introduction

In 1893, George Westinghouse won the contract to provide electric lighting at the Chicago World’s Fair using alternating current (AC), underbidding Thomas Edison’s direct current (DC) proposal by nearly half (Cowdrey, 2006). The event secured the dominance of his and Nikola Tesla’s AC paradigm over Edison’s competing DC architecture. At the time, the voltage transformer offered AC a clear advantage: unlike DC, AC could be transmitted over long distances with minimal loss. As a result, AC became the standard for the generation, transmission, distribution, and consumption of electricity—and remains so 120 years later.

Meanwhile, much has changed about the way society uses electricity. Thirty years ago, most building loads were still AC devices, such as incandescent lamps, heating coils, and induction motors. In today’s buildings, driven by the advent of low-cost computing and advanced power electronics, DC loads are rapidly replacing these traditional loads. A DC load is any device which requires DC electricity either for end-use or as a power conditioning stage—loads such as computers, consumer electronics, cellular telephones, electronic ballast fluorescent lamps, light-emitting diode lamps, and variable frequency motor drives. The interconnection of these DC loads to the AC electric grid requires AC-DC power conversion, introducing energy losses and reducing power system efficiency.

Despite the recent proliferation of DC devices, the design of electricity distribution systems in buildings has changed very little: AC distribution is used throughout the building and AC-DC conversion occurs only at the point of end use—one converter per DC device. In a 21st century revival of the AC versus DC debate, centralized DC electricity distribution systems for buildings have been proposed as a means to reduce the energy losses associated with individual AC-DC converters. DC
distribution has the potential to both increase system efficiency and reduce operating costs (George, 2006; Boroyevich et al., 2010; Savage et al., 2010).

In addition, DC distribution facilitates the integration of DC distributed generation (DG) sources, such as photovoltaic (PV) arrays. The opportunity for energy savings is considerable—by one estimate as much as 8% of the U.S. national electricity consumption (Savage et al., 2010).

Regardless of the apparent efficiency advantage, widespread implementation of building-level DC electricity distribution is presently out of reach for technical and economic reasons. Since whole building DC distribution represents a paradigm shift from traditional AC distribution, significant research is required in order to understand and quantify its potential benefits (George, 2006). Areas of active research include technological barriers (Sannino et al., 2003; Lu and Agelidis, 2009), safety (Engelen et al., 2006; Boroyevich et al., 2010), and system control (Karlsson and Svensson, 2003; Schonberger et al., 2006; Salomonsson and Sannino, 2007).

What is missing from the existing literature, however, is a thorough treatment of the efficiency advantage of DC distribution over AC. Although researchers have examined some aspects of DC distribution efficiency (Sannino et al., 2003; Nilsson and Sannino, 2004; Gentile Polese et al., 2011; AlLee and Tschudi, 2012), no study to date has comprehensively compared the energy efficiency of AC and DC distribution systems for buildings. This article extends the work of Gentile Polese et al. (2011) to provide a framework for such an assessment.

In this article, we present a modeling approach and optimization algorithm to maximize the energy efficiency of electricity distribution systems for commercial buildings via the use of mixed AC-DC electricity distribution. A mixed AC-DC electricity distribution system contains separate AC and DC subsystems which serve AC and DC devices, respectively, reducing losses associated with AC-DC conversion. The major contributions of the article are

1. A mixed-integer, nonlinear, nonconvex mathematical programming model which determines an optimal design of a mixed AC-DC building electricity distribution system, minimizing energy losses,
2. A tailored global optimization algorithm based on Nonconvex Generalized Benders Decomposition (Li et al., 2011), and
3. Illustrative results from three case studies.

The development and reformulation of the mixed AC-DC power flow equations is unique and allows the use of certain reformulation techniques which have not previously applied to power systems analysis. In addition, this article represents the first application of Nonconvex Generalized Benders Decomposition to electric power systems. The research presented in this article extends a dissertation recently submitted by the first author to the Colorado School of Mines (Frank, 2013).

The remainder of the article is organized as follows: in Section 2, we describe the concept of a mixed AC-DC electricity distribution system and review the relevant literature. We outline the mathematical programming formulation in Section 3 and develop an exact decomposition algorithm in Section 4. We present computational results and discussion for the case studies in Section 5 and conclude with Section 6.

2. Problem Description and Literature Review

A building electricity distribution system is the network of power distribution and conversion apparatus which transfers electrical energy between the utility point of common coupling (PCC) (typically, the location of the electric meter) and the point of interconnection (POI) of each electrical load and source within the building. Building electricity distribution systems experience three main categories of energy loss:

1. Resistive (conduction) losses in cables, bus duct, and transformer windings,
2. Magnetic losses in transformer cores, and
3. Conversion losses in AC-DC, DC-AC, and DC-DC power converters.

A rigorous evaluation of mixed AC-DC electricity distribution system efficiency must address losses associated with any piece of equipment whose performance is affected by the choice of AC or DC distribution, including the first energy conversion (if any) at each load or source POI. Fig. 1 illustrates this definition of the system boundaries.

Over a fixed time horizon, the energy efficiency of the distribution system is the ratio of the system input energy $E_{in}$, defined as the total energy delivered at the utility PCC and each source POI,
to the system output energy $E_{\text{Out}}$, defined as total energy consumed by loads at each load POI.

$E_{\text{Loss}} = E_{\text{In}} - E_{\text{Out}}$ is the energy loss associated with electricity distribution and AC-DC conversion. In the mixed AC-DC electricity distribution system design problem considered in this article, the goal is to select a system design which minimizes $E_{\text{Loss}}$.

2.1. Prior Research

Several studies have compared the efficiency of AC and DC distribution systems, but the reported results are conflicting because no single study accounts for all the sources of loss identified in Fig. 1. Although George (2006) and Savage et al. (2010) estimate 10–20% energy savings from the elimination of AC-DC converters (rectifiers) associated with DC loads, neither study addresses other system losses, including losses associated with centralized rectification. Sannino et al. (2003) and Engelen et al. (2006) demonstrate that wiring conduction losses for DC distribution are significantly lower than for AC distribution provided that a sufficiently high DC voltage is selected, but Nilsson and Sannino (2004) conclude that efficiency gains in the wiring may be offset by losses in the centralized AC-DC converters. Lu and Agelidis (2009) present experimental evidence that existing loads operate more efficiently on DC networks than on AC, while AlLee and Tschudi (2012) provide a strong argument for the use of DC distribution in data centers.

With the exception of AlLee and Tschudi (2012), none of these studies formally defines the system boundaries for the efficiency analysis. In addition, the existing research neglects the effect of time-varying loads, providing efficiency estimates for fixed load conditions only. Furthermore, George (2006) and Savage et al. (2010) offer savings projections based on statistical estimation rather than derived from electrical models. Therefore, it is difficult to draw conclusions from the existing literature regarding the system-level energy efficiency of DC distribution. Moreover, prior studies analyze fixed scenarios only: the distribution system is either entirely AC or entirely DC. No study to date has applied an optimization approach to mixed AC-DC electricity distribution system design.

By contrast, the optimization of AC power systems enjoys a rich history. First introduced by Carpentier (1962), the field of optimal power flow (OPF) has become an integral aspect of utility planning and operations. The mixed AC-DC electricity distribution system design problem parallels several interesting and difficult OPF problem classes in power systems planning, including optimal capacitor placement in distribution systems (González et al., 2012), optimal distribution feeder reconfiguration (de Oliveira et al., 2010), and optimal instal-
2.2. System Representation

Electrical power systems may be modeled as networks consisting of a set of buses (nodes) interconnected by a set of branches (arcs). Buses represent physical points of electrical interconnection, such as distribution panels and switchgear, while branches represent paths for power flow: transmission lines, cables, transformers, and similar power systems apparatus. At each bus, electrical power may be supplied by sources or consumed by loads.

Like general network models, electrical networks obey the principle of flow balance, or “conservation of flow” (Rardin, 1997). Conservation of flow requires that for some given quantity (in this case, electrical power), the condition

\[(\text{Total flow out}) - (\text{Total flow in}) = (\text{Supply}) - (\text{Demand}) \quad (1)\]

must hold at every network node. For electrical networks, conservation of flow follows from the application of Kirchoff’s current law.

Conceptually, a mixed AC-DC electricity distribution system may be partitioned into two separate networks: an AC network and a DC network. Each network may contain buses, branches, sources, and loads associated with its type of electricity (AC or DC). AC-DC converters interconnect the two networks, providing a means for electrical power exchange.

The flow balance of an AC network operating at steady state is represented by the matrix equation

\[\tilde{I} = \tilde{Y}\tilde{V}, \quad (2)\]

in which \(\tilde{I}\), \(\tilde{Y}\), and \(\tilde{V}\) are complex quantities that describe the physical behavior of the power system in the frequency domain:

- \(\tilde{I}\) is a vector of AC current injections at each bus in the network, that is, supply minus demand,
- \(\tilde{Y}\) is the system admittance matrix, which models the electrical behavior of the system branches, and
- \(\tilde{V}\) is a state vector of AC bus voltages.

It is more convenient to express (2) in terms of electrical power flow,

\[S = P + jQ = \tilde{V} \circ (\tilde{I}) = \tilde{V} \circ (\tilde{Y}\tilde{V}), \quad (3)\]

in which \(S = P + jQ\) is a vector of complex power injections at each bus in the network, \(j\) represents the imaginary unit \(\sqrt{-1}\), \(\circ\) indicates element-wise multiplication, and the overline operator indicates complex conjugation. Using either the polar or the rectangular representation of \(\tilde{Y}\) and \(\tilde{V}\), (3) may be expanded into a set of nonlinear, real-valued power flow equations. Power systems texts, such as Glover et al. (2013), provide a complete development of the power flow equations and their solution.

The DC power flow equivalent of (3) is

\[P = V \circ (GV), \quad (4)\]

in which

- \(P\) is a vector of DC power injections at each bus,
- \(G\) is the DC system conductance matrix, and
- \(V\) is a state vector of DC bus voltages.

In DC power flow, all quantities are real-valued rather than complex.

A mixed AC-DC electricity distribution system includes separate AC and DC networks linked together by power electronics converters. The power balance equation for such converters is

\[P_{\text{In}} = P_{\text{Out}} + \alpha + \beta P_{\text{Out}} + \gamma P_{\text{Out}}^2, \quad (5)\]

in which \(P_{\text{In}}\) is the converter input power, \(P_{\text{Out}}\) is the output power, and \(\alpha\), \(\beta\), and \(\gamma\) are fitted parameters which model conversion loss. Parameter \(\alpha\) is nonnegative, but \(\beta\) and \(\gamma\) may assume any arbitrary value. Therefore, \(P_{\text{In}}\) is not necessarily a convex nor concave function of \(P_{\text{Out}}\).

Equations (3)–(5) fully describe the behavior of an mixed AC-DC electricity distribution system operating at steady state, and, with suitable transformations, provide a set of real-valued constraints compatible with optimization algorithms.

2.3. Optimal Power Flow

OPF describes the field of research which seeks to optimize the operation of electrical power systems using mathematical programming techniques. OPF differs from other power systems optimization procedures, such as economic dispatch and unit commitment, in that it includes the power flow equations (3) in the formulation. OPF encompasses a wide range of power systems optimization problems and is an extremely active field of research (Frank et al., 2012a,b). Although OPF has traditionally been applied to AC transmission systems, it also
provides a rich foundation for the design and optimization of distribution systems, including mixed AC-DC systems.

2.3.1. Classical Formulation

The classical OPF formulations of Carpentier (1962) and Dommel and Tinney (1968) represent an electrical power system as a set of buses $i \in \mathcal{N}$ which are connected by branches $ik \in \mathcal{B}$. The generators, $\mathcal{G} \subseteq \mathcal{N}$, are located at the buses and operate with a (typically quadratic) cost in their respective real output power: $C_i(P_i^{G})$. Within this framework, we wish to minimize total generation cost subject to operating limits.

The classical formulation is then given by

$$\min \sum_{i \in \mathcal{G}} C_i(P_i^G),$$

subject to

$$P_i(V, \delta) = P_i^G - P_i^L \quad \forall i \in \mathcal{N},$$

$$Q_i(V, \delta) = Q_i^G - Q_i^L \quad \forall i \in \mathcal{N},$$

$$P_i^{G,min} \leq P_i^G \leq P_i^{G,max} \quad \forall i \in \mathcal{G},$$

$$Q_i^{G,min} \leq Q_i^G \leq Q_i^{G,max} \quad \forall i \in \mathcal{G},$$

$$V_i^{min} \leq V_i \leq V_i^{max} \quad \forall i \in \mathcal{N},$$

$$\delta_i^{min} \leq \delta_i \leq \delta_i^{max} \quad \forall i \in \mathcal{N}.$$
nino et al., 2003; Salomonsson and Sannino, 2007; AILee and Tschudi, 2012). In addition, the limited design space of the retrofit case is easier to understand and formulate as a mathematical programming problem than the more open-ended design problem of new construction. Finally, analysis of the retrofit case provides a structured comparison of similarly configured AC, DC, and mixed AC-DC systems: the predefined network layout allows a direct comparison of various types of system losses.

3.1. Framework

The mixed AC-DC electricity distribution system design problem is a nonconvex MINLP of the general form

\[
\begin{align*}
\min & \quad F(x, y), \\
\text{s.t.} & \quad G(x, y) \leq 0, \\
& \quad x \in X, \ y \in Y \text{ and binary.}
\end{align*}
\]

The continuous variables \(x\) model the system voltage states and power flows in each time period. The binary variables \(y\) model design decisions, specifically the allocation of branches, sources, and loads to the AC or DC networks and the inclusion or exclusion of AC-DC, DC-AC, and DC-DC converters. The objective function \(F(x, y)\) is linear and represents the total utility source energy supplied over the specified time horizon. The constraints \(G(x, y)\) include both linear functions and nonconvex, nonlinear functions which represent

1. Power and harmonic current balance constraints at each system bus,
2. Power and harmonic current relationships for branches, sources, loads, and converters,
3. Voltage limits,
4. Converter power and current limits,
5. Restrictions on architecture assignment for branches, sources, and loads,
6. Feasibility restrictions which ensure a properly designed electrical network.

The nonconvexities in the constraints produce a nonconvex feasible region, greatly increasing problem difficulty despite the linear objective function. This design problem has several features in common with SCUC:

- The power flow equations in each time period form the core of the formulation,
- The formulation includes binary decision variables which affect system power flow, and
- The formulation includes several side constraints which govern the feasibility of the design.

Accordingly, the sense of the constraints developed in this article closely parallels existing SCUC formulations (Zhu, 2009). Conversely, the mixed AC-DC electricity distribution system design problem differs from SCUC in that the binary design decisions are independent of the time period. This allows decoupling by time period once the binary decisions are fixed, similar to a two-stage stochastic programming model.

3.2. Notation

Bold italic font, \(a\), denotes decision variables, italic font, \(a\), denotes fixed parameters or indices, and calligraphic font, \(A\), denotes sets. When multiple subscripts are present, we omit the separating commas except in cases in which a numeric value is substituted for a symbol. The network representation in the formulation employs the following indices and sets:

- \(a \in A\): Set of system network architectures \((a = 0 \text{ for DC and } a = 1 \text{ for AC})\),
- \(i, k \in N\): Set of system buses (nodes),
- \(ik \in B\): Set of system branches (arcs),
- \(s \in S\): Set of system sources (supply),
- \(\ell \in L\): Set of system loads (demand),
- \(c \in C\): Set of system converters (AC-DC, DC-AC, or DC-DC),
- \(h \in H\): Set of harmonic frequencies, and
- \(t \in T\): Set of time periods under evaluation.

The set of branches \(B\) exists in \(N \times N\) and branch indices \(i\) and \(k\) denote the buses at which the branch terminates. Sets \(N\) and \(B\) are further partitioned into an AC network containing buses \(N^A\) and branches \(B^A\), and a DC network containing buses \(N^D\) and branches \(B^D\) (Fig. 2).

For harmonics \(h\), harmonic frequency \(f_h = hf_1\), in which \(f_1\) is the fundamental frequency. Therefore, \(h = 0\) corresponds to DC, \(h = 1\) corresponds to fundamental frequency AC, and \(h > 1\) and integer corresponds to the conventional definition of harmonic frequencies in power systems engineering. Binary variables \(x^B\), \(x^S\), \(x^L\), and \(x^C\) model the system design decisions:

- \(x^B_{ik}\) equals 1 if branch \(ik\) is assigned to network architecture \(a\) and 0 otherwise;
- \(x^S_{sa}\) equals 1 if source \(s\) is connected to the system using architecture \(a\), and 0 otherwise;
The objective function does not include energy supplied at harmonic frequencies because the formulation does not directly model power at harmonic frequencies. In general, well-behaved AC sources supply negligible real power at harmonic frequencies. Instead, the power required to support harmonic current injections is drawn from the AC network at the fundamental frequency, with coupling provided through the system loads. The formulation presented in this article does not directly model this coupling, but the fundamental frequency AC load nevertheless indirectly reflects system harmonic losses.

3.3.2. Power and Current Balance

Constraints (9) and (10) enforce the flow balance equation (1) for power and current, respectively. Source and converter output power and current are defined using the source convention (into the bus), while load and converter input power and current are defined using the load convention (out of the bus). Branch power and current are defined as entering the branch (leaving the bus). Therefore, the left hand sides of constraints (9) and (10) represent net power and current injected into each bus and the right hand sides represent net power and current draw out of each bus.

3.3.3. Branch Power and Current

Constraints (11) govern the DC and AC power flow for each system branch. Similarly, constraints (12) govern AC branch current at harmonic frequencies. These constraints are an expanded form of the power flow equations (2)–(4) using rectangular voltage coordinates. (The trigonometric functions which appear in (11) and (12) do not operate on the decision variables but only represent transformed constants.) If branch $ik$ is not physically present in network $a$ ($x_{ika}^B = 0$), then its complex power $P_{ikht}^B + jQ_{ikht}^B$ and current $I_{ikht}^B + jI_{ikht}^B$ must equal zero at each harmonic $h$ and time period $t$; otherwise, the power and current are physical functions of the voltage at each end of the branch.

3.3.4. Source and Load Power and Current

Constraints (13) define the net power for each source and load in the system. Each complex source power $P_{sht}^B + jQ_{sht}^B$ or load power $P_{lht}^B + jQ_{lht}^B$ is a quadratic function of voltage; this type of supply and demand model is common in electrical power systems analysis. As with the branches, the binary
(Objective; See Section 3.3.1)
\[
\min \sum_{\ell \in T} \gamma_\ell \hat{P}_{1,1,\ell}
\]

(Power Balance; See Section 3.3.2)
\[
\text{s.t.} \sum_{\ell \in E} P_{R,0,\ell} - \sum_{\ell \in F} P_{C,0,\ell} - \sum_{\ell \in E} P_{C,0,\ell} = \sum_{k \in B} P_{R,0,k} + \sum_{k \in B} P_{R,0,k} \quad \forall \ell \in N, \forall \ell \in T
\]
\[
\text{s.t.} \sum_{\ell \in E} \hat{P}_{k,1,\ell} - \sum_{\ell \in F} \hat{P}_{k,1,\ell} - \sum_{\ell \in E} \hat{P}_{k,1,\ell} = \sum_{k \in A} P_{R,0,k} + \sum_{k \in A} P_{R,0,k} \quad \forall \ell \in N, \forall \ell \in T
\]

(Current Balance; See Section 3.3.2)
\[
\text{s.t.} \sum_{\ell \in E} \hat{P}_{k,1,\ell} - \sum_{\ell \in F} \hat{P}_{k,1,\ell} - \sum_{\ell \in E} \hat{P}_{k,1,\ell} = \sum_{k \in A} P_{R,0,k} + \sum_{k \in A} P_{R,0,k} \quad \forall \ell \in N, \forall \ell \in T
\]

(Branch Power; See Section 3.3.3)
\[
\text{s.t.} \quad P_{B,0,0}^{ik} = \hat{X}^{ik}_0 \left( \frac{1}{S_{ik,0,0}^{ik} + \frac{S_{ik,0,0}}{2}} \right) V_{k,0}^2 \quad \forall i \in B, \forall \ell \in T
\]
\[
\text{s.t.} \quad P_{B,0,0}^{ik} = \hat{X}^{ik}_0 \left( \frac{S_{ik,0,0}}{2} \right) V_{k,0}^2 \quad \forall i \in B, \forall \ell \in T
\]
\[
\text{s.t.} \quad P_{B,1,1}^{ik} = \hat{X}^{ik}_1 \left( \frac{1}{S_{ik,1,1}^{ik} + \frac{S_{ik,1,1}}{2}} \right) (E_{1,1}^2 + F_{1,1}^2) \quad \forall i \in B, \forall \ell \in T
\]
\[
\text{s.t.} \quad P_{B,1,1}^{ik} = \hat{X}^{ik}_1 \left( \frac{S_{ik,1,1}}{2} \right) (E_{1,1}^2 + F_{1,1}^2) \quad \forall i \in B, \forall \ell \in T
\]
\[
\text{s.t.} \quad Q_{B,1,1}^{ik} = \hat{X}^{ik}_1 \left( \frac{1}{S_{ik,1,1}^{ik} + \frac{S_{ik,1,1}}{2}} \right) (E_{1,1}^2 + F_{1,1}^2) \quad \forall i \in B, \forall \ell \in T
\]
\[
\text{s.t.} \quad Q_{B,1,1}^{ik} = \hat{X}^{ik}_1 \left( \frac{S_{ik,1,1}}{2} \right) (E_{1,1}^2 + F_{1,1}^2) \quad \forall i \in B, \forall \ell \in T
\]
(Branch Current; See Section 3.3.3)

\[\begin{align*}
I_{s,kht}^{B,R} &= x_{ik,1}^{R} \left( \frac{1}{A_{ikht}} \left( g_{ikht}^{s} + g_{ikht}^{sh} \frac{s_{ikht}^{h}}{2} \right) E_{ikht} - \left( b_{ikht}^{s} + b_{ikht}^{sh} \frac{s_{ikht}^{h}}{2} \right) F_{ikht} \right) \\
&+ \frac{1}{A_{ikht}} \left( -g_{ikht}^{s} \cos \varphi_{ikht} + b_{ikht}^{s} \sin \varphi_{ikht} \right) E_{ikht} \\
&+ \frac{1}{A_{ikht}} \left( g_{ikht}^{s} \sin \varphi_{ikht} + b_{ikht}^{s} \cos \varphi_{ikht} \right) F_{ikht} \\
\forall \ i \in B^A, \ h \in H^{D}, \ \forall \ t \in T \\
\end{align*}\]

\[\begin{align*}
I_{s,kht}^{B,\overline{R}} &= x_{ik,1}^{R} \left( \left( g_{ikht}^{s} + g_{ikht}^{sh} \frac{s_{ikht}^{h}}{2} \right) E_{ikht} - \left( b_{ikht}^{s} + b_{ikht}^{sh} \frac{s_{ikht}^{h}}{2} \right) F_{ikht} \right) \\
&+ \frac{1}{A_{ikht}} \left( -g_{ikht}^{s} \cos \varphi_{ikht} - b_{ikht}^{s} \sin \varphi_{ikht} \right) E_{ikht} \\
&+ \frac{1}{A_{ikht}} \left( -g_{ikht}^{s} \sin \varphi_{ikht} + b_{ikht}^{s} \cos \varphi_{ikht} \right) F_{ikht} \\
\forall \ i \in B^A, \ h \in H^{D}, \ \forall \ t \in T \\
\end{align*}\]

\[\begin{align*}
I_{s,kht}^{B,\overline{R},\overline{S}} &= x_{ik,1}^{R} \left( \left( g_{ikht}^{s} + g_{ikht}^{sh} \frac{s_{ikht}^{h}}{2} \right) E_{ikht} + \left( g_{ikht}^{s} + g_{ikht}^{sh} \frac{s_{ikht}^{h}}{2} \right) F_{ikht} \right) \\
&+ \frac{1}{A_{ikht}} \left( -g_{ikht}^{s} \cos \varphi_{ikht} - b_{ikht}^{s} \sin \varphi_{ikht} \right) E_{ikht} \\
&+ \frac{1}{A_{ikht}} \left( -g_{ikht}^{s} \sin \varphi_{ikht} + b_{ikht}^{s} \cos \varphi_{ikht} \right) F_{ikht} \\
\forall \ i \in B^A, \ h \in H^{D}, \ \forall \ t \in T \\
\end{align*}\]

\[\begin{align*}
I_{s,kht}^{S,\overline{R},\overline{S}} &= x_{ik,1}^{R} \left( \frac{1}{A_{ikht}} \left( g_{ikht}^{s} + g_{ikht}^{sh} \frac{s_{ikht}^{h}}{2} \right) E_{ikht} + \left( g_{ikht}^{s} + g_{ikht}^{sh} \frac{s_{ikht}^{h}}{2} \right) F_{ikht} \right) \\
&+ \frac{1}{A_{ikht}} \left( -g_{ikht}^{s} \cos \varphi_{ikht} - b_{ikht}^{s} \sin \varphi_{ikht} \right) E_{ikht} \\
&+ \frac{1}{A_{ikht}} \left( -g_{ikht}^{s} \sin \varphi_{ikht} + b_{ikht}^{s} \cos \varphi_{ikht} \right) F_{ikht} \\
\forall \ i \in B^A, \ h \in H^{D}, \ \forall \ t \in T \\
\end{align*}\]

(Source and Load Power; See Section 3.3.2)

\[\begin{align*}
P_{s,0,0}^{S,R} &= x_{x,0}^{S,R} \left( p_{r,0,0}^{S,R} + g_{r,0,0}^{S,R} f_{r,0,0}^{S,R} - v_{r,0,0}^{S,R} \right) \\
&\forall \ i \in N^{D}, \ s \in S^{D}, \ \forall \ t \in T \\
P_{s,1,1}^{S,R} &= x_{x,1}^{S,R} \left( p_{r,1,1}^{S,R} + g_{r,1,1}^{S,R} f_{r,1,1}^{S,R} - v_{r,1,1}^{S,R} \right) \\
&\forall \ i \in N^{A}, \ s \in S^{A}, \ \forall \ t \in T \\
Q_{s,1,1}^{S,R} &= x_{x,1}^{S,R} \left( q_{r,1,1}^{S,R} - v_{r,1,1}^{S,R} \right) \\
&\forall \ i \in N^{A}, \ s \in S^{A}, \ \forall \ t \in T \\
P_{L,0,0}^{L,R} &= x_{L,0}^{L,R} \left( p_{r,0,0}^{L,R} + g_{r,0,0}^{L,R} f_{r,0,0}^{L,R} + v_{r,0,0}^{L,R} \right) \\
&\forall \ i \in N^{D}, \ \forall \ \ell \in \mathbb{L}^{D}, \ \forall \ t \in T \\
P_{L,1,1}^{L,R} &= x_{L,1}^{L,R} \left( p_{r,1,1}^{L,R} + q_{r,1,1}^{L,R} - v_{r,1,1}^{L,R} \right) \\
&\forall \ i \in N^{A}, \ \forall \ \ell \in \mathbb{L}^{A}, \ \forall \ t \in T \\
Q_{L,1,1}^{L,R} &= x_{L,1}^{L,R} \left( q_{r,1,1}^{L,R} - v_{r,1,1}^{L,R} \right) \\
&\forall \ i \in N^{A}, \ \forall \ \ell \in \mathbb{L}^{A}, \ \forall \ t \in T \\
\end{align*}\]
(Converter Power; See Section 3.3.5)
\[
\begin{align*}
&\text{s.t. } x_c^{C,\min} \leq P_{ch}^C \leq x_c^{C,\max} \\
&\quad x_{ch}^{C,\min} \leq P_{ch}^C \leq x_{ch}^{C,\max} \\
&\quad \sum_{h \in (0,1)} P_{ch}^C = x_c^C \left( \sum_{h \in (0,1)} P_{ch}^C + \alpha_c + \beta_c \sum_{h \in (0,1)} P_{ch}^C + \gamma_c \left( \sum_{h \in (0,1)} P_{ch}^C \right)^2 \right) \\
&\quad x_c^{C,\min} \leq Q_{c,1}^C \leq x_c^{C,\max} \\
&\quad x_{c,1}^{C,\min} \leq Q_{c,1}^C \leq x_{c,1}^{C,\max}
\end{align*}
\]
\(\forall c \in C, \forall t \in T\) (15a)
\(\forall c \in C, \forall h \in (0,1), \forall t \in T\) (15b)
\(\forall c \in C, \forall t \in T\) (15c)
\(\forall c \in C, \forall t \in T\) (15d)
\(\forall c \in C, \forall t \in T\) (15e)

(Converter Current; See Section 3.3.5)
\[
\begin{align*}
&\text{s.t. } -x_{ch}^{C,\max} \leq I_{ch}^C \leq x_{ch}^{C,\max} \\
&\quad -x_{ch}^{C,\max} \leq I_{ch}^C \leq x_{ch}^{C,\max} \\
&\quad -x_{ch}^{C,\max} \leq I_{ch}^C \leq x_{ch}^{C,\max} \\
&\quad -x_{ch}^{C,\max} \leq I_{ch}^C \leq x_{ch}^{C,\max}
\end{align*}
\]
\(\forall c \in C, \forall h \in \mathbb{H}, \forall t \in T\) (16a)
\(\forall c \in C, \forall h \in \mathbb{H}, \forall t \in T\) (16b)
\(\forall c \in C, \forall h \in \mathbb{H}, \forall t \in T\) (16c)
\(\forall c \in C, \forall h \in \mathbb{H}, \forall t \in T\) (16d)

(State Variable Relationships; See Section 3.3.6)
\[
\begin{align*}
&\text{s.t. } \beta_{t,1}^C = E_{t,1}^C + F_{t,1}^C \\
&\quad x_{c,1}^C E_{t}^C \leq E_{t+1}^C \leq x_{c,1}^C E_{t}^C \\
&\quad x_{c,1}^C F_{t}^C \leq F_{t+1}^C \leq x_{c,1}^C F_{t}^C
\end{align*}
\]
\(\forall i \in N^A, \forall t \in T\) (17a)
\(\forall i \in N^A, \forall c \in C_t, \forall h \in \mathbb{H}, \forall t \in T\) (17b)
\(\forall i \in N^A, \forall c \in C_t, \forall h \in \mathbb{H}, \forall t \in T\) (17c)

(Binary Logic; See Section 3.3.7)
\[
\begin{align*}
&\text{s.t. } \sum_{a \in A} x_{a,0}^S = 1 \\
&\quad \sum_{a \in A} x_{a,0}^L = 1 \\
&\quad \sum_{a \in A} x_{a,1}^S = 1 \\
&\quad \sum_{a \in A} x_{a,1}^L = 1 \\
&\quad x_{s,0}^S = x_{s,0}^L + \sum_{k \in B^D} x_{k,0}^B + \sum_{k \in B^D} x_{k,1}^B + \sum_{c \in C^D} x_{c}^C \\
&\quad x_{s,1}^S = x_{s,1}^L + \sum_{k \in B^D} x_{k,0}^B + \sum_{k \in B^D} x_{k,1}^B + \sum_{c \in C^D} x_{c}^C \\
&\quad x_{s,0}^L = x_{s,0}^S + \sum_{k \in B^D} x_{k,0}^B + \sum_{k \in B^D} x_{k,1}^B + \sum_{c \in C^D} x_{c}^C \\
&\quad x_{s,1}^L = x_{s,1}^S + \sum_{k \in B^D} x_{k,0}^B + \sum_{k \in B^D} x_{k,1}^B + \sum_{c \in C^D} x_{c}^C \\
&\quad x_{c}^S = \sum_{s \in S^D} x_{s,0}^S + \sum_{k \in B^D} x_{k,0}^B + \sum_{k \in B^D} x_{k,1}^B \\
&\quad x_{c}^L = \sum_{s \in S^D} x_{s,1}^S + \sum_{k \in B^D} x_{k,0}^B + \sum_{k \in B^D} x_{k,1}^B \\
&\quad x_{c}^S = \sum_{s \in S^D} x_{s,0}^L + \sum_{k \in B^D} x_{k,0}^B + \sum_{k \in B^D} x_{k,1}^B \\
&\quad x_{c}^L = \sum_{s \in S^D} x_{s,1}^L + \sum_{k \in B^D} x_{k,0}^B + \sum_{k \in B^D} x_{k,1}^B \\
&\quad x_{i,0}^S = 0 \\
&\quad x_{i,1}^S = 0 \\
&\quad x_{i,0}^L = 0 \\
&\quad x_{i,1}^L = 0
\end{align*}
\]
\(\forall i \in B \setminus B^D\) (18a)
\(\forall i \in B \setminus B^D\) (18b)
\(\forall i \in B \setminus B^D\) (18c)
\(\forall s \in S\) (18d)
\(\forall s \in S\) (18e)
\(\forall i \in N^D\) (18f)
\(\forall i \in N^D\) (18g)
\(\forall i \in N^D\) (18h)
\(\forall i \in N^D\) (18i)
\(\forall i \in N^D\) (18j)
\(\forall i \in N^D\) (18k)
\(\forall i \in N^D\) (18l)
\(\forall i \in N^D\) (18m)
\(\forall i \in N^D\) (18n)
\(\forall i \in N^D\) (18o)
\(\forall i \in N^D\) (18p)
\(\forall i \in N^D\) (18q)
Instead, the sets $\hat{C}$ include the buses to which the converter terminals connect. Converter losses associated with processing harmonic frequencies do so by constraining the harmonic voltage to zero at the bus where they connect. Constraints (18b) and (18c) enforce this condition for the set of converters $C_i$ at each bus $i$ that are capable of filtering harmonic currents.

### 3.3.6. State Variable Relationships

Constraints (17) enforce relationships among the various state variables. Constraint (17a) is required only at the fundamental frequency because voltage magnitude terms do not appear in the harmonic frequency equations.

Typically, inverters or filters that sink current at harmonic frequencies do so by constraining the harmonic voltage to zero at the bus where they connect. Constraints (17b) and (17c) enforce this condition for the set of converters $C_i$ at each bus $i$ that are capable of filtering harmonic currents.

### 3.3.7. Binary Logic

Constraints (18) enforce several logical conditions that define the feasible design of a mixed AC-DC electricity distribution system. Constraints (18a)–(18c) are packing and partitioning constraints which define the proper assignment of system equipment. Constraints (18a) prohibit a system branch from being assigned to the AC and DC networks simultaneously, although it is permissible for a branch to not be assigned to any network. Constraints (18b) and (18c) ensure that all sources and loads, respectively, are assigned to exactly one network architecture (AC or DC). Because sources and loads represent fixed supply and demand, it is not permissible to omit them from the system. (Otherwise, the optimal solution would include all the sources and omit all the loads.)

Constraints (18d)–(18i) enforce common sense engineering design principles on the network configuration. These constraints are redundant with the power balance equations and definitions for branch,
source, load, and converter power, and may therefore be considered feasibility cuts. The advantage of these cuts is that they operate specifically on the binary variables, tightening the integer feasible region in relaxations of the formulation.

Constraints (18i)–(18k) are optimality cuts for converter placement that ensure that all converter outputs connect to a load or a branch such that there is a destination for power transfer through each converter. Since each converter $c$ incurs fixed loss $\alpha_c$ simply by being present in the system, it is suboptimal to include a converter that serves no load.

Finally, constraints (18l)–(18m), (18n)–(18o), and (18p)–(18q) prevent the assignment of branches, sources, and loads, respectively, from being assigned to an incompatible network architecture.

3.3.8. Variable Bounds

Constraints (19a)–(19c) provide limits on the system voltages. Bounds for $E_{ikht}$ and $F_{ikht}$ are derived from the voltage magnitude limits and any restrictions on the bus voltage angles. Constraints (19d)–(19g) define the scope of the binary decision variables. There are no explicit bounds on the power and current variables.

3.4. Model Reformulation and Strengthening

We employ several reformulation techniques that increase the tractability of the nonconvex MINLP:

1. The isolation of nonconvexity using reformulation variables (Section 3.4.1),

2. Exact linearization of binary-continuous terms using equivalent big $M$ constraints (Section 3.4.2), and

3. Implementation of valid cuts which improve algorithm performance (Section 3.4.3).

The resulting, equivalent mathematical program to (8)–(19) is a nonconvex mixed-integer quadratically constrained programming problem (MIQCP).

3.4.1. Isolation of Nonconvexity

In order to isolate the nonconvexity caused by bilinear terms containing products of continuous variables, we replace such terms with auxiliary reformulation variables $\mathbf{w}$—one per bilinear term. This requires the introduction of reformulation constraints which restrict the value of $\mathbf{w}$ to equal the original bilinear term. For example, the single constraint

$$u = \sum_i \sum_k x_i x_k$$

may be replaced with the set of constraints

$$u = \sum_i \sum_k w_{ik} \quad \text{and} \quad w_{ik} = x_i x_k \quad \forall i, k.$$  

This strategy is part of the Reformulation-Linearization Technique presented in Sherali and Alameddine (1992) and extended in Liberti and Pantelides (2006). The advantages of the reformulation are that it (i) renders the original constraint linear, isolating the nonconvexity in the reformulation constraints, and (ii) facilitates explicit relaxation strategies for the reformulation variables.

Isolation of all continuous bilinear terms in the formulation requires reformulation variables and associated constraints for voltage products (used in the device power definitions) and squared converter power terms (used in the converter input-output power relationship),

$$w_{iht}^{VV} = V_i V_{ht}, \quad \forall i, k \in N \mid i \geq k, \quad \forall h \in \{0, 1\}, \forall t \in T,$$  

$$w_{iht}^{EF} = E_i E_{ht}, \quad \forall i, k \in N \mid i \geq k, \quad \forall h \in \{1\}, \forall t \in T,$$  

$$w_{iht}^{EF} = F_i F_{ht}, \quad \forall i, k \in N, \quad \forall h \in \{1\}, \forall t \in T,$$  

$$w_{ct}^{CC} = \sum_{h \in \{0,1\}} \mathbf{P}_{ch}^{C} \quad \forall c \in C, \forall t \in T,$$  

$$w_{ct}^{CC} = (w_{ct}^{C})^2 \quad \forall c \in C, \forall t \in T.$$  

These $\mathbf{w}$ variables replace their associated bilinear or otherwise composite terms in the formulation. For example, constraint (13a) becomes

$$\mathbf{P}_s = x_{s,0,t} \left( p_{s,0,t}^S + V_i V_{i,t} + s_{i,t} - w_{iht}^{VV} \right).$$

Constraints (11), (13), (15c), and (17a) are all reformulated this way.

3.4.2. Linearization of Binary-Continuous Terms

After reformulating the nonlinearities in the continuous terms, the formulation still contains constraints of the form $u = yf(x)$ where $\mathbf{u}$ and $\mathbf{x}$ are
continuous, \( y \) is binary, and \( f(x) \) is a linear function. Such constraints may be linearized using big \( M \) switching constraints,

\[
\begin{align*}
  f(x) - M^+(1 - y) & \leq u \leq f(x) + M^-(1 - y) \quad (21a) \\
  -M^- y & \leq u \leq M^+ y. \quad (21b)
\end{align*}
\]

When \( y = 0 \), constraints (21a) are nonbinding but constraints (21b) fix the value of \( u \) to zero. When \( y = 1 \), constraints (21b) become nonbinding but constraints (21a) fix the value of \( u \) to \( f(x) \).

We linearize constraints (11) and (13) this way. From (15a), all \( \hat{P}^S_{s,0,t} \leq (P^S_{s,0,t} + V_{i,0,t}^S r^S_{s,0,t} - w^V_{i,0,t} g^S_{s,0,t}) + M^- (1 - x^S_{s,0,t}) \),

\[
\begin{align*}
  \hat{P}^S_{s,0,t} & \geq (P^S_{s,0,t} + V_{i,0,t}^S r^S_{s,0,t} - w^V_{i,0,t} g^S_{s,0,t}) - M^+ (1 - x^S_{s,0,t}) \),
  \hat{P}^S_{s,0,t} & \leq M^+ x^S_{s,0,t}, \quad \hat{P}^S_{s,0,t} \geq -M^- x^S_{s,0,t}.
\end{align*}
\]

Constraint (15c) also follows this pattern, but may be linearized more directly. From (15a), all \( \hat{P}^C_{cht} = 0 \) whenever \( x^C_{c,t} = 0 \). We therefore reformulate (15c) as

\[
\begin{align*}
  \sum_{h \in \{0, 1\}} \hat{P}^C_{cht} = w^C_{ct} + x^C_{c} \alpha_e + \beta_e w^C_{ct} + \gamma_e w^C_{ct}.
\end{align*}
\]

### 3.4.3. Valid Feasibility Cuts

In order to strengthen the model, we introduce two tailored feasibility cuts derived from the power flow equations:

1. Cuts that constrain branch real power losses to be nonnegative, and
2. Cuts that constrain the geometry of the AC voltages.

The cuts are valid in the sense that they hold for any solution which satisfies the original MINLP formulation, that is, the cuts are redundant for the full nonconvex model. However, their presence tightens the feasible region in mixed-integer linear programming (MILP) relaxations of the model.

The branch loss cuts take the form

\[
P^B_{ik} + P^B_{ki} \geq 0, \quad (22)
\]

which states that the real power entering the branch from each terminal must sum to a nonnegative net power which represents the branch loss. The reasoning is that branch losses can never be negative, as then the branch would be creating energy—a violation of the laws of physics. Similar cuts may be derived for branch reactive power loss, but the sense of the inequality depends on the electrical characteristics of the branch. Unlike real power, reactive power represents circulating energy, and therefore may either be “lost” in an inductive branch or “gained” in a capacitive branch.

The voltage magnitude cuts follow from the definition of complex AC voltage in rectangular coordinates,

\[
\tilde{V} = E + jF.
\]

Real and imaginary voltage components \( E \) and \( F \) are orthogonal, as illustrated in Fig. 3. From the geometry of the voltage triangle and the trigonometric relationship

\[
V^2 = E^2 + F^2,
\]

the cuts

\[
|V| \leq |E| + |F| \quad \text{and} \quad (23)
\]

\[
|E| \leq |V| \quad \text{and} \quad (24)
\]

\[
|F| \leq |V| \quad \text{and} \quad (25)
\]

are valid for all feasible values of \( V, E \), and \( F \). In practice, cuts (23)–(25) require either a linearization of the absolute value function or a restriction on the signs of \( V, E \), and \( F \). Therefore, we apply the cuts only to voltage combinations for which the variable bounds enable dropping the absolute value function.

### 4. Tailored Decomposition Algorithm

After reformulation, the mixed AC-DC electricity distribution system design problem is a MIQCP of
the general form
\[
    z = \min_{x, y} F(x, y), \\
    \text{s.t.} \quad G(x, y) \leq 0, \quad (P) \\
    x \in X, \quad y \in Y \text{ and integer.}
\]

The objective function \( F \) is linear. The constraints \( G \) are nonlinear and nonconvex, but are separable in \( x \) and \( y \). In addition, \( (P) \) decomposes in \( x \) by time period when \( y \) is fixed, although it remains nonconvex.

Current state-of-the-art global optimization solver cannot efficiently solve a monolith for this problem (see Section 5). However, the two-stage structure of \((P)\) lends itself to efficient decomposition techniques. In this section, we present an efficient solution algorithm for \((P)\) based on Nonconvex Generalized Benders Decomposition (NGBD).

Under certain mild assumptions, namely that the feasible integer solution spaces of both \((P)\) and its convex relaxation are bounded, the algorithm guarantees both finite convergence and a globally optimal solution of \((P)\) within a finite tolerance.

The tailored algorithm includes two preprocessing techniques which improve the tractability of the formulation: computation of minimal big \( M \) values and a bound-tightening procedure. We also develop two enhancements to the NGBD algorithm which decrease solution time without adversely affecting solution quality.

### 4.1. Nonconvex Generalized Benders Decomposition

Generalized Benders Decomposition (GBD) (Geoffrion, 1972) is a decomposition procedure used to solve nonlinear problems of the form
\[
    \min_{x, y} F(x, y) \\
    \text{s.t.} \quad G(x, y) \leq 0, \quad (26) \\
    x \in X, \quad y \in Y,
\]
in which \( y \) is a set of “complicating” variables in the sense that \((26)\) becomes much easier to solve for \( x \) when \( y \) is fixed. Although GBD does not require that \((26)\) be convex jointly in \( x \) and \( y \), it does require that functions \( F \) and \( G \) are convex in \( x \) when \( y \) is fixed. Therefore, GBD cannot be applied to the mixed AC-DC electricity distribution system design problem.

NGBD (Li et al., 2011) extends GBD to nonlinear, nonconvex problems of the form
\[
    \min_{\xi, y} \sum_{t \in T} F_t(\xi_t, y) \\
    \text{s.t.} \quad G_t(\xi_t, y) \leq 0, \quad \forall t \in T, \quad (27) \\
    \xi_t \in \Xi_t, \quad \forall t \in T, \quad y \in Y \text{ and integer},
\]
in which \( T \) represents a set of time periods or scenarios which are independent for fixed \( y \). Problem \((27)\) is a special case of \((P)\) in which
\[
    x = \{\xi_1, \ldots, \xi_{|T|}\}, \\
    F(x, y) = \sum_{t \in T} F_t(\xi_t, y), \\
    G(x, y) = G_1(\xi_1, y) \times \ldots \times G_{|T|}(\xi_{|T|}, y), \\
    X = \Xi_1 \times \ldots \times \Xi_{|T|}.
\]

Since \((P)\) and \((27)\) are equivalent for the mixed AC-DC electricity distribution system design problem, we use the more compact notation of \((P)\) throughout the article.

Like GBD, NGBD seeks solutions to \((P)\) by generating trial points for \( y \) and computing the corresponding feasible solutions \( x \) using separate but related optimization problems. The lower bounding problem
\[
    z = \min_{x, y} F(x, y), \\
    \text{s.t.} \quad G(x, y) \leq 0, \quad (LBP) \\
    x \in X, \quad y \in Y \text{ and integer},
\]
is a convex relaxation of \((P)\) that generates trial points \( y = y'_t \) and associated lower bounds \( z_i \) on the optimal objective function value of \((P)\). In \((LBP)\), \( F \) and \( G \) are convex underestimators for \( F \) and \( G \), respectively.

The upper bounding problem
\[
    \tau(y) = \min_{x} F(x, y), \\
    \text{s.t.} \quad G(x, y) \leq 0, \quad (UBP) \\
    x \in X,
\]
is a nonconvex NLP in \( x \) for fixed \( y \). Given a trial point \( y = y'_t \), solving \((UBP)\) yields a feasible solution \( x = x'_t \) and associated upper bound \( \tau_t \) on the optimal objective function value of \((P)\).

The main loop of the NGBD algorithm solves \((P)\) by iteration between \((LBP)\) and \((UBP)\) (Fig. 4).
Figure 4: Nonconvex Generalized Benders Decomposition algorithm.
The logic behind NGBD is as follows: the algorithm solves a sequence of increasingly restrictive (LBP)s, generating a sequence of non-decreasing lower bounds $\bar{z}_i$, each of which applies to the current trial point $y = y'_i$ and all feasible solutions $y$ that have not yet been visited. Then, by solving a sequence of (UBP)s using the trial points $y'_i$ at each iteration $i$, the algorithm generates a sequence of upper bounds $\bar{z}_i$. These upper bounds are not necessarily non-increasing nor non-decreasing; therefore the algorithm must track which is the best. At each iteration $i$, the tightest upper bound $\bar{z}_i$ corresponds to the best feasible solution $(x'_i, y'_i)$ found during any iteration up to and including $i$. As soon as the lower bound $\bar{z}_i$ on the *unvisited* values of $y$ is greater than or equal to the upper bound $\bar{z}_i$ on the *visited* values of $y$ within tolerance $\varepsilon$, then no value of $y$ will yield a better solution than the current best solution $(x'_i, y'_i)$. Therefore, at algorithm termination, $(x'_i, y'_i)$ is globally optimal within tolerance $\varepsilon$.

After each iteration, the algorithm adds the solution elimination constraint $y \neq y'_i$ to (LBP). The solution elimination constraints ensure that the main loop never generates a given trial point more than once. In the worst case, NGBD enumerates all points $y \in Y$, solving (UBP) at each trial point. The finite convergence of NGBD therefore requires that domain $Y$ contain a finite number of integer points. As described by Li et al. (2011) and Li et al. (2012a), the full NGBD algorithm uses GBD to solve (LBP) at each iteration. However, any solution procedure for (LBP) may replace GBD provided that (i) it is finitely convergent and (ii) it returns a global optimum to (LBP). In order for NGBD to guarantee an $\varepsilon$-optimal solution, the optimality tolerances for the (LBP) and (UBP) algorithms must sum to $\varepsilon$. In our tailored NGBD algorithm (Section 4.5), we solve the (LBP) monolith rather than employ a decomposition algorithm.

### 4.2. Convex Relaxation

Both NGBD and the bound-tightening procedure require a convex relaxation of (P). Strictly speaking, all mixed-integer programs (linear or nonlinear) are nonconvex due to the integrality requirements. Nevertheless, problem (7) may be termed a convex MINLP if the functions $F$ and $G$ are convex. Convex MINLPs are significantly more tractable than nonconvex MINLPs because they are readily solved to global optimality using local NLP solvers coupled with branch and bound techniques on the integer variables.

The convex MINLP $(\bar{P})$ is a convex relaxation of a nonconvex MINLP $(P)$ if (i) every feasible solution for $(P)$ is also feasible for $(\bar{P})$, and (ii) the optimal objective function value $\bar{z}^*$ of $(\bar{P})$ is better than or the same as the optimal objective function value $z^*$ of $(P)$. Solving convex relaxation $(\bar{P})$ provides a lower bound on the objective function value of $(P)$.

After reformulation, the nonconvexities in the mixed AC-DC electricity distribution system design problem are isolated in quadratic reformulation constraints of the form $w = x^2$ and bilinear reformulation constraints of the form $w = x_1x_2$. Therefore, a convex relaxation for the problem may be obtained by replacing these reformulation constraints with convex underestimators.

#### 4.2.1. Relaxation of Bilinear Terms

To obtain the relaxation of (P), we first create the McCormick inequalities (McCormick, 1976) to the definitions of the reformulation variables $w_{kkh}, w_{kh}, w_{khh}, w_{khh}, \text{ and } w_{khh}$ (20a)-(20d).

#### 4.2.2. Relaxation of Univariate Quadratic Terms

The quadratic term $w = x^2$ is a special case of the bilinear term $w = x_1x_2$ in which $x_1 = x_2 = x$. If $x \in [x^L, x^U]$, then the McCormick inequalities

\begin{align}
 w & \geq 2x^Lx - (x^L)^2, \quad (28a) \\
 w & \geq 2x^Ux - (x^U)^2, \quad (28b) \\
 w & \leq x^Lx + x^Ux - x^Lx^U \quad (28c)
\end{align}

provide a valid linear relaxation of $w$. However, relaxation (28) is generally weak unless $x^L$ and $x^U$ are close together.

Because the function $f(x) = x^2$ is convex, the underestimating terms (28a)-(28b) may be strengthened by adding cuts at points intermediate to $x^L$ and $x^U$. For a univariate, differentiable, convex function $w = f(x)$, the tangent line at any point $x = x'$ provides the valid underestimator which is tight at $x = x'$,

\[ w \geq f(x') + \frac{df}{dx}|_{x=x'}(x - x'). \]

Therefore, given a set of $K$ values of $x'$ such that $x^L \leq x'_k \leq x^U$ for $k \in \{1, \ldots, K\}$, the set of inequalities

\begin{align}
 w & \geq 2x'_kx - (x'_k)^2 \quad \forall k \in \{1, \ldots, K\}, \quad (29a) \\
 w & \leq x^Lx + x^Ux - x^Lx^U, \quad (29b)
\end{align}
provide a valid linear relaxation that is tighter than (28). Given a sufficiently large K, the underestimating constraints (29a) can be made arbitrarily tight. However, the overestimating constraint (29b) remains weak. We apply this convex relaxation strategy to the definition of the reformulation variables \( w_{\text{CC}}^{\text{FF}} \) (20f) as well as to \( w_{\text{ikh},t}^{\text{FF}}, w_{\text{ikh},k}^{\text{FF}} \), and \( w_{\text{ikh},t}^{\text{CC}} \) for cases in which \( i = k \).

Remark 1. Because the function \( f(x) = x^2 \) is convex, the underestimating terms (28a)–(28b) may instead be replaced with the exact expression \( w \geq x^2 \) to produce a convex MIQCP. However, this would change the problem class from a MILP to a convex MIQCP. Testing indicated that the improvement in relaxation tightness did not justify the large increase in solution time for the instances of the design problem examined in this article.

4.3. Computation of Big M Values

The use of arbitrarily large values of \( M \) in switching constraints (21) can lead to slow convergence or numerical instability in practical MILP solvers. It is therefore desirable to use the smallest possible values for \( M^+ \) and \( M^- \) that provide adequate switching for \( u \).

Given simple bounds \( x \in X \) for the continuous variables, the big \( M \) pre-processing procedure solves the auxiliary problems

\[
M^+ = \max \left\{ \max_{x \in X} f(x), 0 \right\}
\]

and

\[
M^- = \max \left\{ -\min_{x \in X} f(x), 0 \right\}
\]

for each big \( M \) constraint set in (P). Setting \( M = 0 \) if it would otherwise be negative simplifies the construction of (21) while retaining the correct sense of the constraints, facilitating automatic elimination of redundant constraints by MILP solvers.

4.4. Bound Tightening

Bound tightening procedures can improve the performance of both conventional branch-and-bound algorithms, which branch on integer variables, and spatial branch-and-bound algorithms, which branch by partitioning the domains of the continuous decision variables. Bound tightening restricts the problem domain and, by extension, reduces the size of the branch-and-bound tree. Although many nonconvex MINLP solvers employ internal bound tightening, performing bound tightening explicitly during pre-processing allows exploitation of special structure and can further improve algorithm performance.

We perform bound tightening using a convex relaxation \( \ddot{P} \) for problem (P) instead of the original MINLP. Since the feasible domain of (P) is by definition a subspace of the feasible domain of \( \ddot{P} \), any feasibility-based bound tightening that is valid for \( \ddot{P} \) remains valid for (P).

Given a convex relaxation \( \ddot{Q} \) of the nonconvex constraints \( G \) in (P), the lower bound tightening problem

\[
z_k^L = \min \ x_k \text{ or } y_k,
\]

s.t.

\[
\ddot{Q}(x, y) \leq 0,
\]

\[
x \in X, \ y \in Y,
\]

and upper bound tightening problem

\[
z_k^U = \max \ x_k \text{ or } y_k,
\]

s.t.

\[
\ddot{Q}(x, y) \leq 0,
\]

\[
x \in X, \ y \in Y,
\]

compute valid bounds for each variable: \( z_k^L \leq x_k \leq z_k^U \) for continuous variables or \( [z_k^L] \leq y_k \leq [z_k^U] \) for integer variables. For computational speed, the integer decisions \( y \) are relaxed to be continuous during bound tightening, hence the need for floor and ceiling operations to obtain the tightened bounds.

Problems (LBT) and (UBT) may be solved sequentially for each continuous variable \( x \) and integer variable \( y \) to perform bound tightening over the entire domain of (P). In addition, knowledge of the exact nonlinear relationships in (P) may be exploited to improve the tightness of the relaxation. For example, consider the generic reformulation variable \( w_{ik} \) which is the product of reformulation variables \( x_i \in [L_i, U_i] \) and \( x_k \in [L_k, U_k] \). After any update to the bounds on \( x_i \) or \( x_k \), the bounds on \( w_{ik} \) may be tightened according to

\[
w_{ik}^L \leftarrow \max \left\{ w_{ik}^L, \min \left\{ x_i^L x_k^L, x_i^U x_k^L, x_i^L x_k^U, x_i^U x_k^U \right\} \right\},
\]

\[
w_{ik}^U \leftarrow \min \left\{ w_{ik}^U, \max \left\{ x_i^L x_k^L, x_i^U x_k^L, x_i^L x_k^U, x_i^U x_k^U \right\} \right\}.
\]

Similarly, for the generic reformulation variable \( u_{ii} = x_i^2 \), an update to upper bound \( u_{ii}^U \) may be used to tighten the upper and lower bounds on \( x_i \),

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The loop repeats until the bounds converge within numerical error. The iterative portion of the bound tightening algorithm operates on the nonlinear and reformulation variables only, since nonlinear relationships may be exploited to improve their bounds. (Without exploitation of nonlinear relationships, bound tightening provides no further improvement after the first iteration because (LBT) and (UBT) are linear programs for this particular design problem.) The harmonic power flow constraints for $h > 1$ decouple as separate linear programs. There- fore, the (LBP) solution for harmonic currents $h > 1$ is exact and (UBP) need model only DC ($h = 0$) and fundamental frequency AC ($h = 1$), for which the power flow constraints remain nonlinear.

Remark 2. Although the binary variables could be included in the iterative portion of the bound tightening procedure, testing indicated that the gains are minimal for the instances of the design problem examined in this article.

4.5. Algorithm

After computing big $M$ values and performing bound tightening, we generate problems (LBP) and (UBP) and solve (P) using a variant of the NGBD algorithm presented in Section 4.1. The tailored algorithm includes enhancements to address instances of poor performance in the (LBP) and (UBP).

4.5.1. Solution to Lower Bounding Problem

For the mixed AC-DC electricity distribution system design problem, (LBP) follows from (P) by replacing all reformulation constraints for the $w$ variables with their convex relaxations as described in Section 4.2. The resulting mathematical program is a MILP and may be solved directly using commercial mixed-integer linear solvers.

Alternatively, if the number of scenarios is large, Benders Decomposition may be used to solve the (LBP), as in the full NGBD algorithm. Benders Decomposition applies because (LBP) exhibits the necessary block structure: the problem decouples into individual, linear Benders subproblems in $x$ when binary variables $y$ are fixed. Since subproblems are linear, they produce exact duality-based feasibility and optimality cuts as required for the Benders master problem in $y$.

4.5.2. Solution to Upper Bounding Problem

Because $y$ is fixed, the (UBP) is in general a continuous, nonconvex NLP. For the mixed AC-DC electricity distribution system design problem in particular, it is a continuous, nonconvex, quadratically-constrained program (QCP) with a linear objective function. Problem (UBP) has several characteristics which aid its tractability:

- (UBP) decouples by time period into distinct subproblems (UBP)$_t$.
- The harmonic power flow constraints for $h > 1$ decouple as separate linear programs. Therefore, the (LBP) solution for harmonic currents $h > 1$ is exact and (UBP) need model only DC ($h = 0$) and fundamental frequency AC ($h = 1$), for which the power flow constraints remain nonlinear.
- Since $y$ is fixed, there is no need for big $M$ reformulations in (UBP); exact equality constraints may be used instead. This significantly reduces the number of constraints per time period compared to (LBP).

For each time period $t \in T$, subproblem (UBP)$_t$ may be solved using a global optimization algorithm; aggregation of the solutions for each subproblem generates the solution to (UBP).

In order to verify that NGBD converges to an $\varepsilon$-optimal solution, the (UBP) solver employed must compute a globally valid lower bound. This limits the field of potential solvers to global, nonconvex,
Figure 5: Bound tightening procedure for the mixed AC-DC electricity distribution system design problem.
nonlinear solvers, such as BARON, LINDOGlobal, and GloMIQO. Although local nonlinear solvers can often provide solutions of comparable quality in less time, they do not provide a globally valid lower bound. If local nonlinear solvers or heuristic optimization methods are used to solve (UBP), then NGBD no longer guarantees a global optimum.

4.5.3. Algorithmic Enhancements

In practice, the NGBD algorithm may exhibit poor performance (slow convergence or a lack of progress) in either the (LBP) or the (UBP) during any given iteration. To address poor performance in the (LBP), we introduce a solution generation time limit. To ensure that the lower bounds generated by the (LBP) remain valid, we alter the algorithm to set \( \bar{z}_i \) to the lower bound on the (LBP) solution at each iteration rather than to the objective function value. If the (LBP) solver is unable to determine an optimal solution within the allotted time, the algorithm then transfers the incumbent (LBP) solution \( y = y'_i \) to the (UBP) regardless of the gap. This enhancement generates trial points more rapidly but with a possible reduction in trial point quality.

Note that \( \bar{z}_i \) remains a true, but weaker, lower bound for (P). In addition, because the NGBD termination criterion now references (LBP) lower bounds, this enhancement has the effect that the (LBP) tolerance no longer penalizes the final optimality gap of the NGBD algorithm.

To address poor performance in the (UBP), we also introduce a time limit for solving each (UBP) subproblem. For each subproblem, the algorithm then tracks both the partial objective function value \( \zeta_{it} \) such that

\[
\bar{z}_i := \sum_{t \in T} \zeta_{it}
\]

and the final subproblem gap \( \delta_{it} \) such that

\[
\epsilon'_i := \sum_{t \in T} \delta_{it}
\]

represents the (UBP) optimality gap for iteration \( i \).

This (UBP) enhancement sacrifices the a priori guarantee of an \( \varepsilon \)-optimal solution in order to secure an upper time limit for each NGBD iteration. Nevertheless, upon successful algorithmic termination it is possible to determine if the solution obtained is \( \varepsilon \)-optimal. First, if \( \epsilon'_i \leq \varepsilon \) for each iteration \( i \), then the best solution \((x'_i, y'_i)\) is \( \varepsilon \)-optimal. Even if some \( \epsilon'_i > \varepsilon \), however, solution \((x'_i, y'_i)\) is still \( \varepsilon \)-optimal if two weaker conditions are met:

1. To ensure that \((x'_i, y'_i)\) is \( \varepsilon \)-optimal for the (UBP), it is required that \( \epsilon'_i \leq \varepsilon \).
2. For all other iterations \( i \neq i' \), the relationship

\[
\bar{z}_i - \epsilon'_i \geq \bar{z}_{i'} - \varepsilon
\]

must be satisfied. This ensures that no other (UBP) examined could have yielded a solution to (P) that is better than the optimum obtained at iteration \( i' \) by more than \( \varepsilon \).

Together, these conditions ensure that solution \((x'_i, y'_i)\) is globally optimal for (P) within tolerance \( \varepsilon \). If the conditions are not satisfied, then the offending (UBP) subproblems may be resolved without time limits, if desired. Otherwise, the final optimality gap is

\[
\bar{z}_{i'} - \min_i \{ \bar{z}_i - \epsilon'_i \}.
\]

5. Computational Results

We present computational results for three case studies. The first is a three bus system constructed to test the capabilities of the algorithm and does not represent a typical building. The second and third cases model a typical three-story office building under reduced and full DC distribution retrofit scenarios, respectively. Data for both systems are available in Frank (2013).

We implemented the NGBD algorithm for the mixed AC-DC electricity distribution system design problem using the General Algebraic Modeling System (GAMS) (Rosenthal, 2012). Source code and test cases are available in the online appendix of EJOR. All tests were performed on dual six core Intel Xeon X5670 CPUs operating at 2.93 GHz and 48 GB of RAM running GAMS 24.1.3 (64-bit) on Ubuntu Linux 12.04.4. Each test used a single core. The (LBP) solver used was CPLEX (International Business Machines, Inc., 2009): 12.5.1.0 for the three bus and full medium office building test cases, and 12.6.0.0 for the reduced medium office building test case (due to numerical difficulties with 12.5.1.0). The global solvers used were BARON 12.5.0 (Tawarmalani and Sahinidis, 2005), GloMIQO 2.3 (Floudas and Misener, 2013), and LindoGlobal 8.0.1283.385 (Lin and Schrage, 2009).

5.1. Three Bus Test Case

The three bus test system (Fig. 6a) contains three buses, three branches, two sources, and two
loads. Although building electricity distribution systems typically have radial topologies, this test system uses a loop topology to test the capabilities of the solution algorithm: the loop topology allows a large number of feasible designs with similar performance characteristics.

The design decisions include the allocation of the PV array, the lighting load, and the three branches to the AC or DC network and the inclusion or exclusion of two active converters. The utility source and the compressor motor load require AC connections. The test case examines three frequencies—DC ($h = 0$), the AC fundamental ($h = 1$), and the 3rd harmonic ($h = 3$)—and includes three time periods.

Prior to presolve, the MIQCP formulation for the test case contains 12 binary variables, 341 continuous variables, and 871 constraints. There are 108 possible design combinations, of which 28 are feasible. Although the problem is of a modest size, the nonconvexity and tightly constrained feasible regions present significant challenges for state-of-the-art global solvers. In contrast, the (LBP) relaxation contains 334 variables (12 binary) and 1477 constraints—a trivial size for modern MILP solvers.

### 5.1.1. Algorithm Performance

Table 1 displays the performance for the global solvers BARON, LINDOGlobal, and GLoMIQO for the MIQCP monolith for the three bus test case. All three solvers terminated upon reaching a 24 hour (86400 second) time limit. BARON was unable to find a feasible solution. GLoMIQO and LINDOGlobal returned solutions identical to the optimum found by NGBD, but their optimality could not be verified to within a 1.0% tolerance within the time limit.

In contrast, NGBD solves the test case rapidly to within 1.0%. Table 2 displays the performance of the NGBD algorithm with and without bound tightening using BARON, GLoMIQO, and LINDOGlobal as the (UBP) solver. NGBD is orders of magnitude faster than solving the monolith: with or without bound tightening, NGBD using either BARON or GLoMIQO returned a globally optimal solution in under 30 seconds. LINDOGlobal was unable to solve some (UBP) subproblems to the specified tolerance within a 10 minute (600 second) time limit. Nevertheless, because of the enhancements described in Section 4.5.3, the algorithm is still able to guarantee a global optimum within 1.0% upon termination.

Bound tightening accelerates NGBD convergence, reducing the number of NGBD iterations required by a factor of 4. However, because of the modest problem size, the added overhead of bound tightening offsets the reduction in NGBD iteration time, increasing total execution time for BARON and GLoMIQO.

Fig. 7 plots the convergence of the NGBD algorithm (with BARON) for the test case, with and without bound tightening. The convergence plots for GLoMIQO and LINDOGlobal appear very similar. The algorithm finds the optimal design early in the search: on the first iteration with bound tightening and on fourth without. Without bound tightening, the algorithm visits 26 of 28 feasible designs; bound tightening reduces the number of designs visited to only 7.

### 5.1.2. Results

Fig. 6b displays the optimal design for the three bus test case. The final design places the PV array and lighting load in the DC network and connects...
Table 1: Solver performance for the monolith for the three bus test case.

<table>
<thead>
<tr>
<th>Bound tightening</th>
<th>No</th>
<th>Yes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solver</td>
<td>BARON</td>
<td>GLoMIQO</td>
</tr>
<tr>
<td>Total time (s)</td>
<td>86400</td>
<td>86400</td>
</tr>
<tr>
<td>Objective (kWh)</td>
<td>-1.796</td>
<td>1.795</td>
</tr>
<tr>
<td>Lower bound (kWh)</td>
<td>1.791</td>
<td>1.774</td>
</tr>
<tr>
<td>Final gap (%)</td>
<td>∞</td>
<td>1.23%</td>
</tr>
</tbody>
</table>

Table 2: NGBD algorithm performance for the three bus test case.

<table>
<thead>
<tr>
<th>Bound tightening</th>
<th>No</th>
<th>Yes</th>
</tr>
</thead>
<tbody>
<tr>
<td>(UBP) solver</td>
<td>BARON</td>
<td>GLoMIQO</td>
</tr>
<tr>
<td>NGBD iterations</td>
<td>26</td>
<td>26</td>
</tr>
<tr>
<td>Big M time (s)</td>
<td>1.7</td>
<td>1.7</td>
</tr>
<tr>
<td>Bound tightening time (s)</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(LBP) time (s)</td>
<td>4.0</td>
<td>4.1</td>
</tr>
<tr>
<td>(UBP) time (s)</td>
<td>4.8</td>
<td>5.1</td>
</tr>
<tr>
<td>Total time (s)</td>
<td>13.2</td>
<td>13.7</td>
</tr>
<tr>
<td>Objective (kWh)</td>
<td>1.796</td>
<td>1.796</td>
</tr>
<tr>
<td>Lower bound (kWh)</td>
<td>1.778</td>
<td>1.778</td>
</tr>
<tr>
<td>Final gap (%)</td>
<td>0.99%</td>
<td>1.00%</td>
</tr>
</tbody>
</table>

† Solver terminated after 10 minute time limit for some (UBP) subproblems.

5.2. Reduced Medium Office Building Test Case

The United States Department of Energy maintains a database of commercial reference building models (CRBM) which represent typical commercial buildings of various end-use types. The models are intended to provide a baseline for comparison among competing building technologies. Collectively, the available models represent approximately 70% of commercial building stock in the United States (Field et al., 2010; Torcellini et al., 2008).

The CRBM medium office building is an archetypical three story office building with exterior dimensions of 50 m × 33.3 m. The total interior floor space is 4982 m² (1661 m² per floor). Waters et al. (2014) present a standard electricity distribution system for this building for the Seattle location and the 1980–2000 age category. This distribution system provides an ideal case study for the energy efficiency of retrofit DC distribution systems.

Fig. 8 presents the layout of the building electricity distribution system. The boxed labels indicate panel and subpanel designations per (Waters et al., 2014). The system includes one main panel each at 480 V and 208 V. Each floor has subpanels and load circuits for interior lighting, electric heating, and receptacle loads. Dedicated circuits exist for parking lot lighting, elevators, rooftop HVAC equipment, and a PV array.

Both the reduced and full medium office building test cases examine six total frequencies: DC \((h = 0)\), the AC fundamental \((h = 1)\), and odd harmonics \(h = 3\) through \(h = 9\). The model includes load and generation profiles for four typical weekdays—one per season. Each day is partitioned into six time periods.

The reduced medium office building test case examines design choices for only the 208 V network and the rooftop panel HMR. The design decisions include

- Allocation of receptacle and rooftop HVAC loads to either the AC or the DC network,
- Allocation of the PV array to either the AC or the DC network,
- Allocation of selected cables to either the AC or the DC network,
• Inclusion or exclusion of the 112.5 kVA step-down transformer, and

• Inclusion or exclusion of AC-DC converters serving panels HMR and LDP.

The remaining portions of the network are fixed to AC.

Prior to presolve, the MIQCP formulation for the reduced test case has 31920 variables (48 binary) and 87932 constraints. The (LBP) has 31297 variables (48 binary) and 107565 constraints. The (UBP) subproblems are considerably smaller: 592 continuous variables and 776 constraints each.

5.2.1. Algorithm Performance

For the monolith of this test case, with or without bound tightening, BARON terminates after approximately 7 hours without finding a feasible solution, while GLoMIQO terminates with an internal error during presolve. With or without bound tightening, the final lower bound computed by BARON prior to termination was 266 MWh: a gap of 66% compared to the optimum computed by NGBD. (LINDOGlobal was not tested.)

In contrast, NGBD solves the test case to a gap of 1.0%. Table 3 displays the performance of the NGBD algorithm with and without bound tightening using BARON and GLoMIQO. (LINDOGlobal was omitted from the tests due to poor performance in the three bus test case.) For this test case, the (UBP) subproblems are sufficiently difficult that neither GLoMIQO nor BARON is able to verify infeasibility for some of the trial points within the 30 minute (1800 second) subproblem time limit. As a result, with bound tightening, the algorithm spends approximately 88% of its total time on the final iteration when using BARON and 23% when using GLoMIQO. Although BARON solves feasible subproblems an order of magnitude faster than GLoMIQO, GLoMIQO tends to identify infeasible subproblems more quickly. Since infeasibility detection dominates the solution time, the total time using GLoMIQO for this test case is less than when using BARON.

Bound tightening reduces the number of NGBD iterations by 33%. More importantly, bound tightening avoids the need to visit many infeasible solutions prior to termination, that is, it eliminates many iterations with particularly poor (UBP) solver performance. As a result, bound tightening reduces total solution time by approximately 95%.
Figure 8: CRBM medium office building electricity distribution system.
Table 3: NGBD algorithm performance for the reduced medium office building test case.

<table>
<thead>
<tr>
<th>Bound Tightening</th>
<th>No</th>
<th>Yes</th>
</tr>
</thead>
<tbody>
<tr>
<td>(UBP) Solver</td>
<td>BARON</td>
<td>GLoMIQO</td>
</tr>
<tr>
<td>NGBD iterations</td>
<td>70</td>
<td>70</td>
</tr>
<tr>
<td>Big M time (s)</td>
<td>139</td>
<td>139</td>
</tr>
<tr>
<td>Bound tightening time (s)</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(LBP) time (s)</td>
<td>2427</td>
<td>2379</td>
</tr>
<tr>
<td>(UBP) time (s)</td>
<td>459047†</td>
<td>387448†</td>
</tr>
<tr>
<td>Total time (s)</td>
<td>46427</td>
<td>391368</td>
</tr>
<tr>
<td>Objective (MWh)</td>
<td>783.9</td>
<td>783.9</td>
</tr>
<tr>
<td>Lower bound (MWh)</td>
<td>776.2</td>
<td>776.1</td>
</tr>
<tr>
<td>Final gap (%)</td>
<td>0.99%</td>
<td>1.00%</td>
</tr>
</tbody>
</table>

In addition, it significantly improves GLoMIQO solution times for feasible (UBP) subproblems (Fig. 9). However, bound tightening has a minor effect on BARON solution times and a slight adverse effect on (LBP) solution times. Fig. 10 plots the convergence of the NGBD algorithm (with BARON) for the test case, with and without bound tightening; the convergence plots for GLoMIQO are similar. The NGBD algorithm finds the optimal design on the fourth iteration with bound tightening and on the tenth without. However, in both cases the algorithm enumerates all 46 feasible solutions prior to termination.

5.2.2. Results

The optimal solution for the reduced medium office building test case converts the 208 V network to DC and eliminates the step-down transformer, but retains panel HMR and its connected devices in the AC network (Fig. 11). This design represents an 0.7% reduction in utility source energy and a 3% reduction in energy loss compared to a baseline case which uses only AC distribution.

5.3. Full Medium Office Building Test Case

The full medium office building test case adds decisions for the remaining portion of the 480 V network:

- Allocation of elevator and lighting loads to either the AC or the DC network,
- Allocation of the remaining cables to either the AC or the DC network, and
- Inclusion or exclusion of AC-DC converters serving panels MHDP, HM1, HL1, HL2, and HL3.

Figure 9: Distribution of GLoMIQO (UBP) solution times for feasible subproblems in the reduced medium office building test case: (a) without bound tightening, (b) with bound tightening.
At 31950 variables (77 binary) and 87952 constraints, the MIQCP formulation for the full test case is similar in overall size to the reduced test case but contains 60% more discrete variables. The (LBP) has 31326 variables (77 binary) and 107584 constraints. The (UBP) subproblems are identical to the reduced test case: 592 continuous variables and 776 constraints each.

5.3.1. Algorithm Performance

As with the reduced test case, BARON is unable to find a feasible solution to the monolith for the full test case and GLoMIQO terminates with an internal error during presolve. The final lower bound computed by BARON was 71.1 MWh with bound tightening and 70.7 MWh without—a gap of approximately 91% compared to the NGBD solution. NGBD determines a high quality solution, but is unable to close the gap to within 1.0% within a 14 day (1209600 second) time limit. Table 4 displays the performance of the NGBD algorithm with and without bound tightening using BARON and GLoMIQO. Although bound tightening decreases the final gap significantly and improves average GLoMIQO (UBP) solution times, it also increases average (LBP) solution time per iteration by approximately 130%. As a result, NGBD completes approximately half as many iterations with bound tightening as without it.

Fig. 12 plots the convergence of the NGBD algorithm (with BARON) for the test case, with and without bound tightening. The algorithm finds its best solution very early, but progress toward closing the gap is slow due to the large number of feasible trial points generated by the (LBP). The convergence characteristics for GLoMIQO are similar but cover fewer iterations. Although the NGBD algorithm cannot prove optimality within 1.0%, the general trend of the solutions suggests that the solution obtained is nevertheless of high quality.

5.3.2. Results

The best solution found for the full test case is identical to the optimal solution for the reduced test case (Fig. 11): retain the 480 V network as AC and convert the 208 V network to DC.

5.4. Discussion

The mixed AC-DC electricity distribution system design problem leads to very large scale nonconvex MIQCPs for real-world applications; the cur
Figure 11: Optimal solution for reduced medium office building test case.
Table 4: NGBD algorithm performance for the full medium office building test case.

<table>
<thead>
<tr>
<th>Bound Tightening</th>
<th>No</th>
<th>Yes</th>
</tr>
</thead>
<tbody>
<tr>
<td>(UBP) Solver</td>
<td>BARON</td>
<td>GLoMIQO</td>
</tr>
<tr>
<td>NGBD iterations</td>
<td>375</td>
<td>284</td>
</tr>
<tr>
<td>Big M time (s)</td>
<td>139</td>
<td>139</td>
</tr>
<tr>
<td>Bound tightening time (s)</td>
<td>-</td>
<td>3833</td>
</tr>
<tr>
<td>(LBP) time (s)</td>
<td>1194714</td>
<td>842552</td>
</tr>
<tr>
<td>(UBP) time (s)</td>
<td>8034</td>
<td>368433‡</td>
</tr>
<tr>
<td>Total time (s)</td>
<td>1211182‡</td>
<td>1213326‡</td>
</tr>
<tr>
<td>Objective (MWh)</td>
<td>783.9</td>
<td>783.9</td>
</tr>
<tr>
<td>Lower bound (MWh)</td>
<td>737.1</td>
<td>736.3</td>
</tr>
<tr>
<td>Final gap (%)</td>
<td>5.98%</td>
<td>6.07%</td>
</tr>
</tbody>
</table>

† Solver terminated after 30 minute time limit for some (UBP) subproblems.
‡ Algorithm terminated after 14 day time limit.

Figure 12: Convergence of NGBD algorithm (with BARON) for full medium office building test case: (a) without bound tightening, (b) with bound tightening.

rent state-of-the-art global solvers can compute neither feasible solutions nor tight bounds for these models. In contrast, the tailored NGBD algorithm returns very good distribution system designs and computes tight bounds. The algorithm is able to determine globally optimally solutions for moderately sized design problems and still provides good solutions for larger design problems.

Nevertheless, the case studies illuminate several algorithmic challenges. Because the objective function values for competing designs lie close together, tight gap settings are required to prove global optimality. However, even after bound tightening, the (LBP) relaxation is loose compared to the differences in objective function value among competing designs. As a result, the algorithm tends to enumerate most or all feasible integer solutions prior to termination. This challenge results from the unique nature of OPF: OPF problems typically have very tight feasible regions, and, since losses represent only a small fraction of total demand, a very tight range of objective function values. For example, the objective function values of the best and worst feasible solutions for the three bus test case differ by only 0.026 kWh (1.4%). Although the differences in objective function value are numerically small, they are nevertheless significant: for the three bus test case, a 1.4% difference in total energy consumption represents an 15% reduction in electrical loss.

The other challenges pertain to solver performance. As problem size increases, the (LBP) solution time per iteration is large and increases as the (LBP) becomes more restrictive. Use of a (LBP) decomposition algorithm, such as Benders Decomposition, may alleviate this issue. In addition, solution times for the (UBP) are variable and occasion-
ally very poor. However, the algorithmic enhancements described in Section 4.5.3 allow the NGBD algorithm to recover when progress stalls for a particular (UBP) subproblem.

Despite these challenges, the performance of the NGBD algorithm for the mixed AC-DC electricity distribution system design problem is comparable to NGBD performance for other nonconvex engineering design problems (Li et al., 2011, 2012b; Chen et al., 2012). NGBD solution times scale well despite the significantly larger dimensionality of the case studies examined in this article. Although the algorithm tends to enumerate a larger portion of the feasible region, a majority of the trial points generated yielded feasible solutions. Therefore, the total number of iterations required for convergence remains similar to the engineering case studies presented in Li et al. (2011).

6. Conclusions

In this article, we present a mathematical formulation and tailored solution algorithm for the design of energy efficient mixed AC-DC electricity distribution systems for commercial buildings. We describe a unique MIQCP formulation and relaxation strategy which allow the design problem to be solved using efficient decomposition techniques, develop preprocessing techniques to determine efficient big M values, present a bound tightening procedure, and propose an enhanced NGBD algorithm which allows recovery in the event of difficult subproblems.

The proposed algorithm efficiently solved two of three design test cases and found a high quality solution for the third. In contrast, state-of-the-art global solvers encountered difficulty in solving the test cases, neither finding feasible solutions nor computing tight bounds. The case study results demonstrate that the proposed algorithm can efficiently optimize small mixed AC-DC electricity distribution systems and can still suggest high quality designs for larger design cases.

The tightly constrained nature of the design problem does present challenges for the proposed algorithm, namely slow progress in solving the relaxation (LBP) for the larger test cases and the tendency of the algorithm to enumerate all feasible solutions prior to termination. Promising avenues for future research include the application of a decomposition technique, such as Benders Decomposition, to the (LBP) to improve performance and the application of enhanced cuts or dual information to reduce the optimality gap, as proposed by Chen et al. (2012).

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References

Engelen, K., Shun, E., Vermeyen, P., Pardon, I., D’hulst, D., Dommel, H., Tinney, W., 1968. Optimal power flow solutions prior to termination. Promising avenues for future research include the application of a decomposition technique, such as Benders Decomposition, to the (LBP) to improve performance and the application of enhanced cuts or dual information to reduce the optimality gap, as proposed by Chen et al. (2012).

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